Uncertainty Traps

Pablo Fajgelbaum[1] Edouard Schaal[2]
Mathieu Taschereau-Dumouchel[3]

[1]UCLA [2]New York University

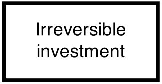
[3]Wharton School
University of Pennsylvania

September 4-5, 2014
University of Cambridge
Aggregate Demand, the Labor Market and Macroeconomic Policy

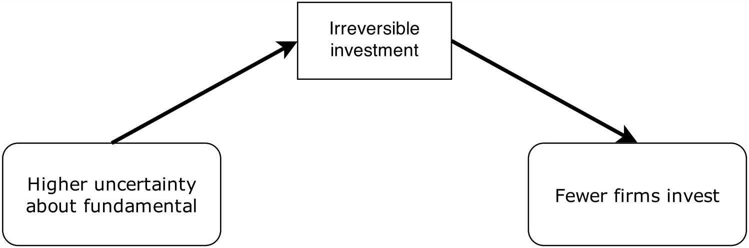
Introduction

- Some recessions are particularly persistent
  - Slow recoveries of 1990-91, 2001
  - Recession of 2007-09: output, investment and employment still below trend Details
- Persistence is a challenge for standard models of business cycles
  - Measures of standard shocks typically recover quickly
    - TFP, financial shocks, volatility...
  - Need strong propagation channel to transform short-lived shocks into long-lasting recessions
- We develop a business cycles theory of endogenous uncertainty
  - Large evidence of heightened uncertainty in 2007-2012 (Bloom et al.,2012; Ludvigson et al.,2013)

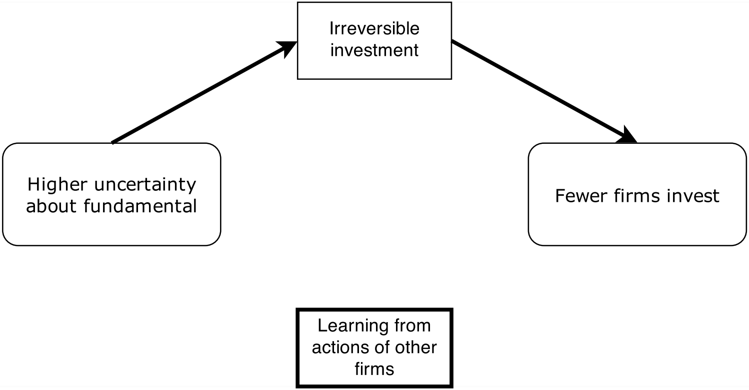
Mechanism



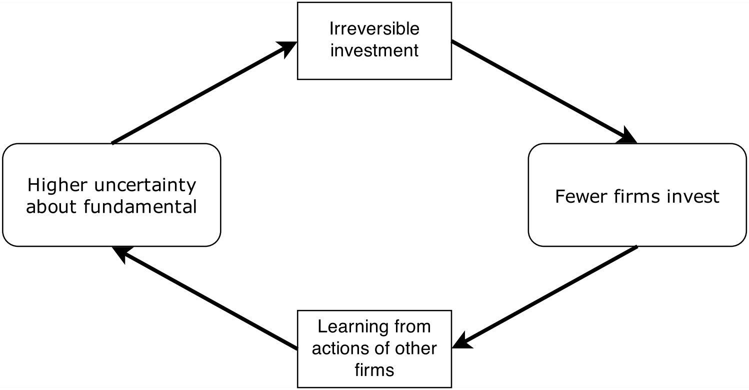
Mechanism


Irreversible investment
Higher uncertainty about fundamental
Fewer firms invest

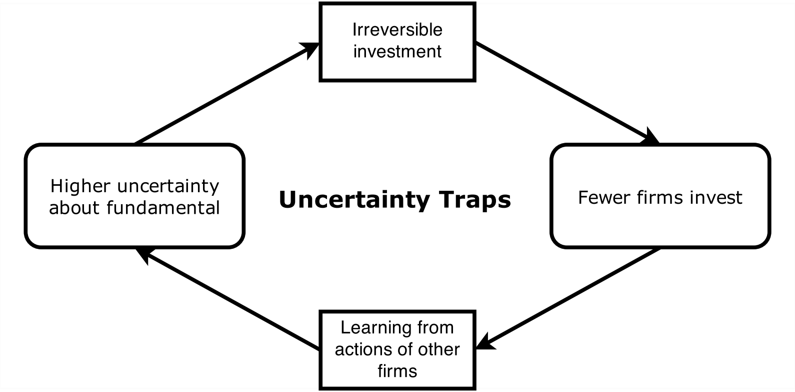
Mechanism


Irreversible investment
Higher uncertainty about fundamental
Fewer firms invest
Learning from actions of other firms

Mechanism


Irreversible investment
Fewer firms invest
Learning from actions of other firms
Higher uncertainty about fundamental

Mechanism



- **Uncertainty traps:**
  - Self-reinforcing episodes of high uncertainty and low economic activity

Roadmap

- Start with a stylized model
  - Isolate how key forces interact to create uncertainty traps
    - Complementarity between economic activity and information strong enough to sustain multiple regimes
  - Establish conditions for their existence, welfare implications
- Extend the model to more standard RBC environment
  - Compare an economy with and without endogenous uncertainty
  - The mechanism generates substantial persistence

▸ Evidence

Theoretical Model

- Infinite horizon model in discrete time
- $\overline{N}$ atomistic firms indexed by $n \in \{1, \ldots, \bar{N}\}$ producing a homogeneous good
- Firms have CARA preferences over wealth

$$u(x) = \frac{1}{a}\left(1 - e^{-ax}\right)$$

Investment and Adjustment Costs

- Each firm n has a *unique* investment opportunity and must decide to either do the project today or wait for the next period
  - Firms face a random fixed investment cost $f \sim$ cdf F, iid, with variance σ^f
  - $N \in \{1, \cdots, \overline{N}\}$ is the endogenous number of firms that invest.
  - Firms that invest are immediately replaced by firms with new investment opportunities
- The project produces output

$$x_n = \theta + \varepsilon_n^x$$

  - Aggregate productivity (the **fundamental**) θ follows a random walk

$$\theta' = \theta + \varepsilon^\theta$$

  and $\varepsilon^\theta \sim \text{iid}\,\mathcal{N}\left(0, \gamma_\theta^{-1}\right)$, $\varepsilon_n^x \sim \text{iid}\,\mathcal{N}\left(0, \gamma_x^{-1}\right)$.

Investment and Adjustment Costs

- Each firm n has a *unique* investment opportunity and must decide to either do the project today or wait for the next period
  - Firms face a random fixed investment cost $f \sim$ cdf F, iid, with variance σ^f
  - $N \in \{1, \cdots, \overline{N}\}$ is the endogenous number of firms that invest.
  - Firms that invest are immediately replaced by firms with new investment opportunities
- The project produces output

$$x_n = \theta + \varepsilon_n^x$$

  - Aggregate productivity (the **fundamental**) θ follows a random walk

$$\theta' = \theta + \varepsilon^\theta$$

and $\varepsilon^\theta \sim \text{iid}\,\mathcal{N}\left(0, \gamma_\theta^{-1}\right)$, $\varepsilon_n^x \sim \text{iid}\,\mathcal{N}\left(0, \gamma_x^{-1}\right)$.

Information

Firms do not observe θ directly, but receive noisy signals:

1. Public signal that captures the information released by media, agencies, etc.

$$Y = \theta + \varepsilon^y, \text{ with } \varepsilon^y \sim \mathcal{N}\left(0, \gamma_y^{-1}\right)$$

2. Output of **all** investing firms
   - Each individual signal

$$x_n = \theta + \varepsilon_n^x, \text{ with } \varepsilon_n^x \sim \text{iid}\, \mathcal{N}\left(0, \gamma_x^{-1}\right)$$

   can be summarized by the aggregate signal:

$$X \equiv \frac{1}{N}\sum_{n \in I} x_n = \theta + \frac{1}{N}\sum_{n \in I} \varepsilon_n^x \sim \mathcal{N}\left(0, (N\gamma_x)^{-1}\right)$$

- Note:
  - No bounded rationality: firms use all available information efficiently
  - No asymmetric information

Timing

Each firm starts the period with common beliefs

1. Firms draw investment cost f and decide to invest or not
2. Production takes place, public signals X and Y are observed
3. Agents update their beliefs and θ' is realized

Beliefs and Uncertainty

- Before observing signals, firms share the same beliefs about θ

$$\theta|\mathcal{I} \sim \mathcal{N}\left(\mu, \gamma^{-1}\right)$$

- Our notion of uncertainty is captured by the variance of beliefs $1/\gamma$
  - Subjective uncertainty, as perceived by decisionmakers, crucial to real option effects
  - Time-varying risk or volatility (Bloom et al., 2012) is a special case

Law of Motion for Beliefs

- After observing signals X and Y, the *posterior about θ* is

$$\theta \quad | \quad \mathcal{I}, X, Y \sim \mathcal{N}\left(\mu_{post}, \gamma_{post}^{-1}\right)$$

with

$$\mu_{post} = \frac{\gamma\mu + \gamma_y Y + N\gamma_x X}{\gamma + \gamma_y + N\gamma_x}$$
$$\gamma_{post} = \gamma + \gamma_y + N\gamma_x$$

- Next period's *beliefs about $\theta' = \theta + \varepsilon^\theta$* is

$$\mu' = \mu_{post}$$
$$\gamma' = \left(\frac{1}{\gamma_{post}} + \frac{1}{\gamma_\theta}\right)^{-1} \equiv \Gamma\left(N, \gamma\right)$$

Firm Problem

- Firms choose whether to invest or not

$$V(\mu,\gamma,f) = \max \left\{ \underbrace{V^W(\mu,\gamma)}_{\text{wait}}, \underbrace{V^I(\mu,\gamma) - f}_{\text{invest}} \right\}$$

- Decision is characterized by a threshold $f_c(\mu,\gamma)$ such that

$$\text{firm invests} \Leftrightarrow f \leq f_c\left(\mu,\gamma\right)$$

Firm Problem

- Value of waiting

$$V^W(\mu,\gamma) = \beta\mathbb{E}\left[\int V\left(\mu',\gamma',f'\right) dF\left(f'\right) \mid \mu,\gamma\right]$$

with $\mu' = \frac{\gamma\mu+\gamma_y Y+N\gamma_x X}{\gamma+\gamma_y+N\gamma_x}$ and $\gamma' = \Gamma\left(N,\gamma\right)$

- Value of investing

$$V^I(\mu,\gamma) = \mathbb{E}\left[u\left(x\right)|\mu,\gamma\right]$$

Aggregate Consistency

- The aggregate number of investing firms N is

$$N = \sum_n \mathbb{1}\left(f_n \leq f_c\left(\mu, \gamma\right)\right)$$

- Firms have the same ex-ante probability to invest

$$p\left(\mu, \gamma\right) = F\left(f_c\left(\mu, \gamma\right)\right)$$

- The number of investing firms follows a binomial distribution

$$N\left(\mu, \gamma\right) \sim Bin\left[\bar{N}, p\left(\mu, \gamma\right)\right]$$

Recursive Equilibrium

Definition

An equilibrium consists of the threshold $f_c(\mu, \gamma)$, value functions $V(\mu, \gamma, f)$, $V^W(\mu, \gamma)$ and $V^I(\mu, \gamma)$, and a number of investing firms $N(\mu, \gamma, \{f_n\})$ such that

1. The value functions and policy functions solve the Bellman equation;
2. The number of investing firms N satisfies the consistency condition;
3. Beliefs (μ, γ) follow their laws of motion.

Characterizing the Evolution of Beliefs: Mean

- Mean beliefs μ follow

$$\mu' = \frac{\gamma\mu + \gamma_y Y + N\gamma_x X}{\gamma + \gamma_y + N\gamma_x}$$

Lemma

For a given N, mean beliefs μ follow a random walk with time-varying volatility s,

$$\mu'|\mu, \gamma = \mu + s(N, \gamma)\varepsilon,$$

with $\frac{\partial s}{\partial N} > 0$ and $\frac{\partial s}{\partial \gamma} < 0$ and $\varepsilon \sim \mathcal{N}(0, 1)$.

Characterizing the Evolution of Beliefs: Precision

- Precision of beliefs γ follow

$$\gamma' = \Gamma(N, \gamma) = \left(\frac{1}{\gamma + \gamma_y + N\gamma_x} + \frac{1}{\gamma_\theta}\right)^{-1}$$

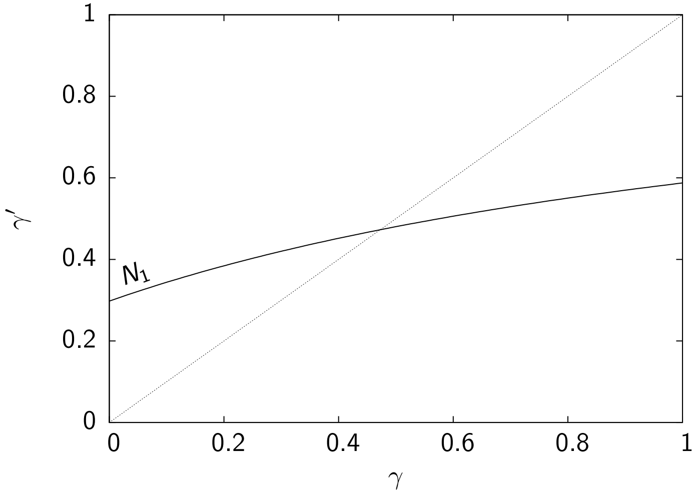
Lemma

1) Belief precision γ' increase with N and γ,
2) For a given N, $\Gamma(N, \gamma)$ admits a unique stable fixed point in γ.

Characterizing the Evolution of Beliefs

- Precision of beliefs γ follow

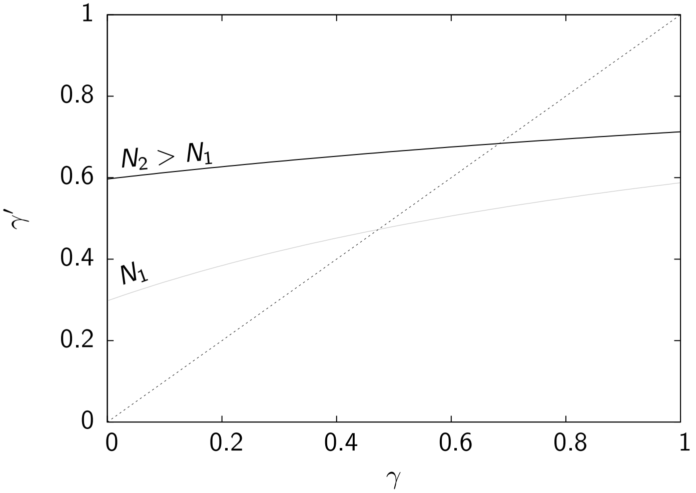
$$\gamma' = \Gamma(N, \gamma)$$



Characterizing the Evolution of Beliefs

- Precision of beliefs γ follow

$$\gamma' = \Gamma(N, \gamma)$$



Equilibrium Characterization

Proposition

Under some weak conditions and for γ_x small,

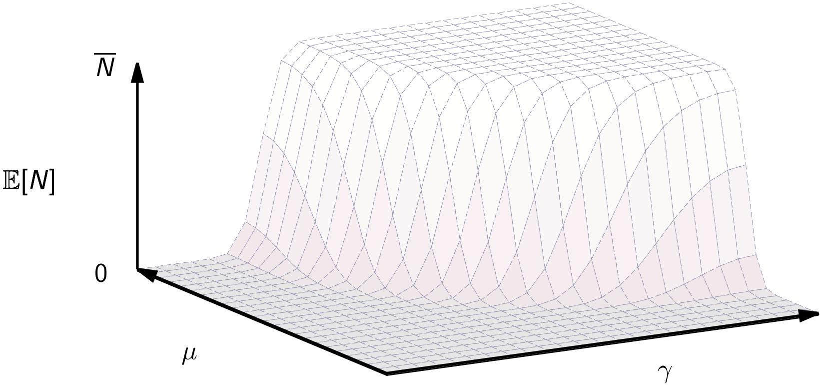
1) The equilibrium exists and is unique;
2) The investment decision of firms is characterized by the cutoff $f_c(\mu, \gamma)$ such that:

$$\textit{firm with cost } f \textit{ invests} \Leftrightarrow f \leq f_c(\mu, \gamma)$$

3) f_c is a strictly increasing function of μ and γ.

Conditions

Aggregate Investment Pattern



Uncertainty Traps

- We now examine the existence of uncertainty traps
  - Long-lasting episodes of high uncertainty and low economic activity
- We now take the limit as $\bar{N} \to \infty$,

$$\frac{N}{\bar{N}} = F\left(f_c\left(\mu, \gamma\right)\right)$$

▸ Details

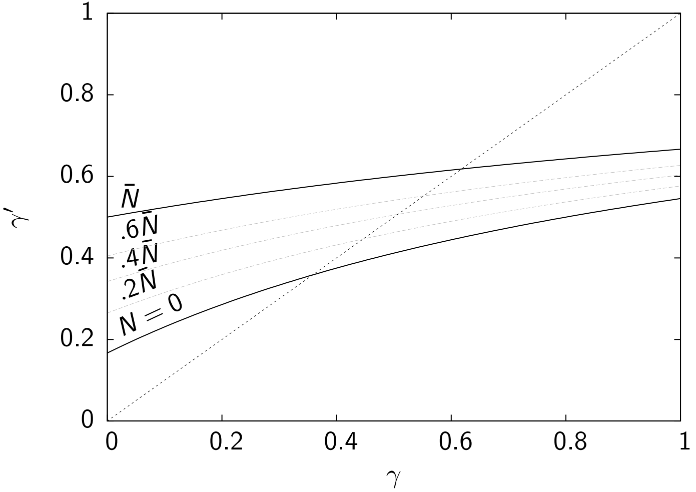
- The whole economy is described by the two-dimensional system:

$$\begin{cases} \mu' & = \mu + s\left(N\left(\mu, \gamma\right), \gamma\right)\varepsilon \\ \gamma' & = \Gamma\left(N\left(\mu, \gamma\right), \gamma\right) \end{cases}$$

Equilibrium Dynamics of Belief Precision

- Precision of beliefs γ follow

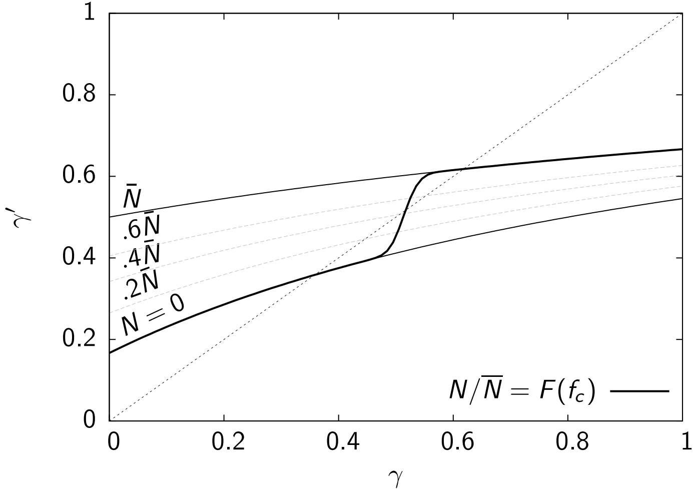
$$\gamma' = \Gamma(N, \gamma)$$



Equilibrium Dynamics of Belief Precision

- Precision of beliefs γ follow

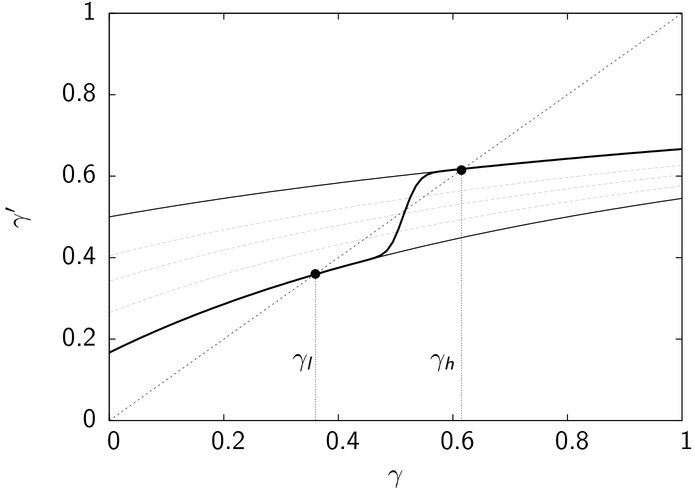
$$\gamma' = \Gamma\left(N\left(\mu, \gamma\right), \gamma\right)$$



Equilibrium Dynamics of Belief Precision

- Precision of beliefs γ follow

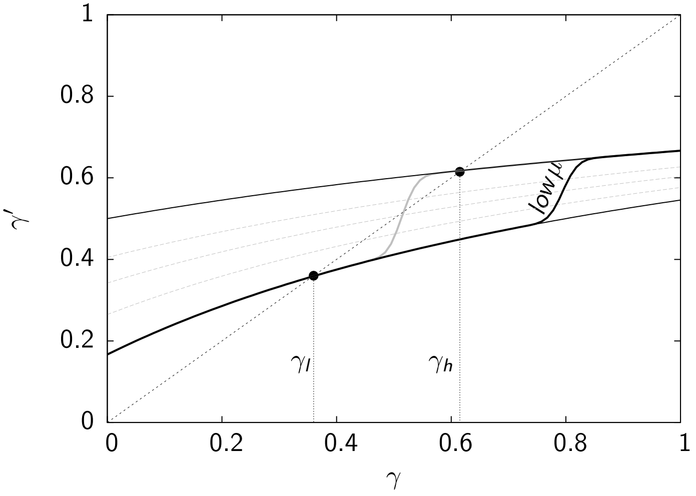
$$\gamma' = \Gamma\left(N\left(\mu,\gamma\right),\gamma\right)$$



Equilibrium Dynamics of Belief Precision

- Precision of beliefs γ follow

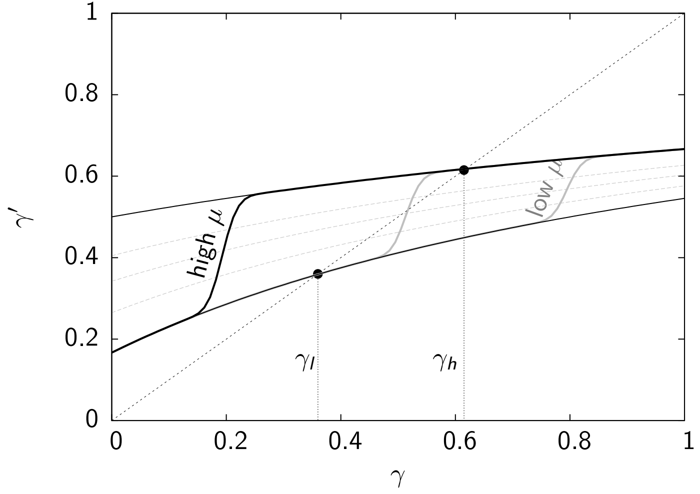
$$\gamma' = \Gamma\left(N\left(\mu, \gamma\right), \gamma\right)$$



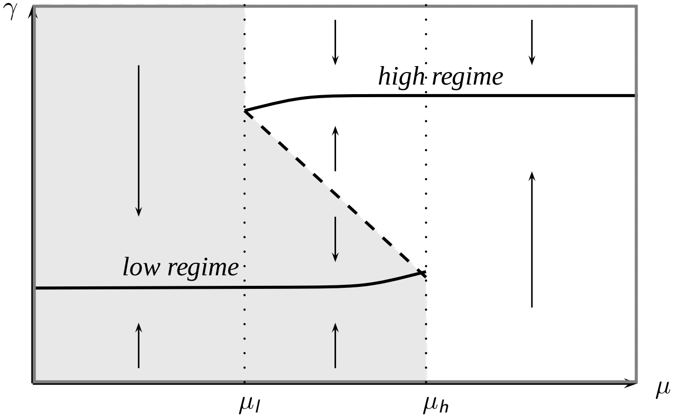
Equilibrium Dynamics of Belief Precision

- Precision of beliefs γ follow

$$\gamma' = \Gamma\left(N\left(\mu, \gamma\right), \gamma\right)$$



Phase diagram



Existence of Uncertainty Traps

Definition

Given mean beliefs μ , there is an uncertainty trap if there are at least two locally stable fixed points in the dynamics of beliefs precision $\gamma' = \Gamma\left(N\left(\mu,\gamma\right),\gamma\right)$.

- Does not mean that there are multiple equilibria
  - The equilibrium is unique,
  - The past history of shocks determines which regime prevails

Existence of Uncertainty traps

Proposition

For γ_x *and* σ^f *low enough, there exists a non-empty interval* $[\mu_l, \mu_h]$ *such that, for all* $\mu_0 \in (\mu_l, \mu_h)$*, the economy features an uncertainty trap with at least two stable steady states* $\gamma_l(\mu_0) < \gamma_h(\mu_0)$*. Equilibrium* γ_l *(*γ_h*) is characterized by high (low) uncertainty and low (high) investment.*

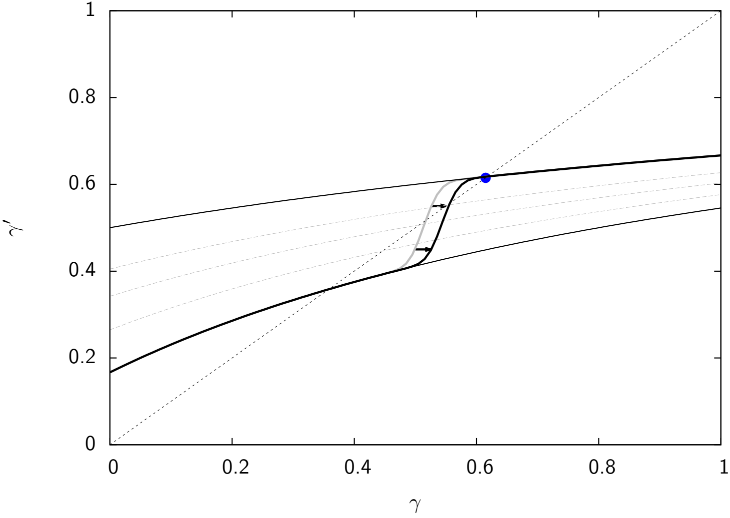
- The dispersion of fixed costs σ^f must be low enough to guarantee a strong enough feedback from information on investment

Uncertainty Traps: Falling in the Trap

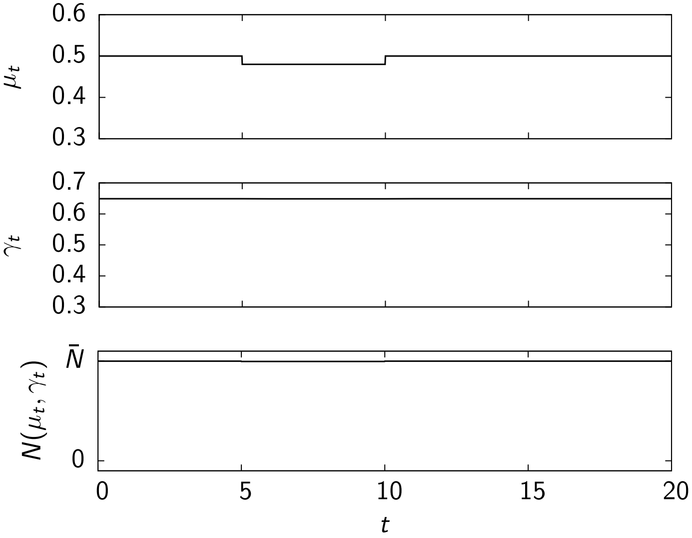
- We now examine the effect of a negative shock to μ
  - Economy starts in the high regime
  - Hit the economy at $t = 5$ and last for 5 periods
  - We consider small, medium and large shocks
- Under what conditions can the economy fall into an uncertainty trap?

Uncertainty Traps: Falling in the Trap

Impact of a **small** negative shock to μ

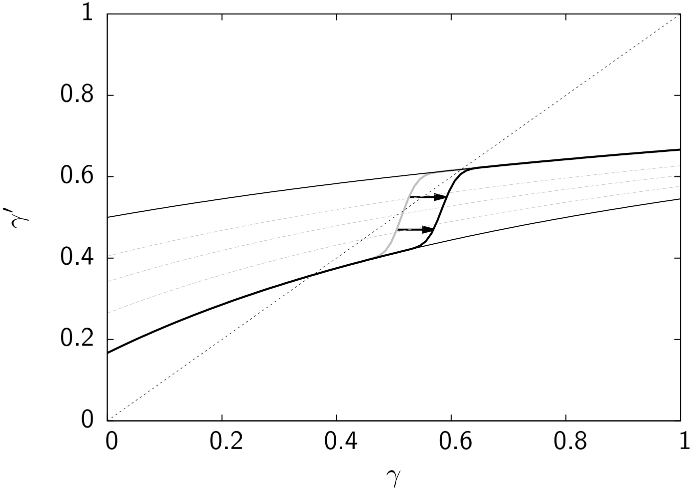


Uncertainty Traps: Falling in the Trap



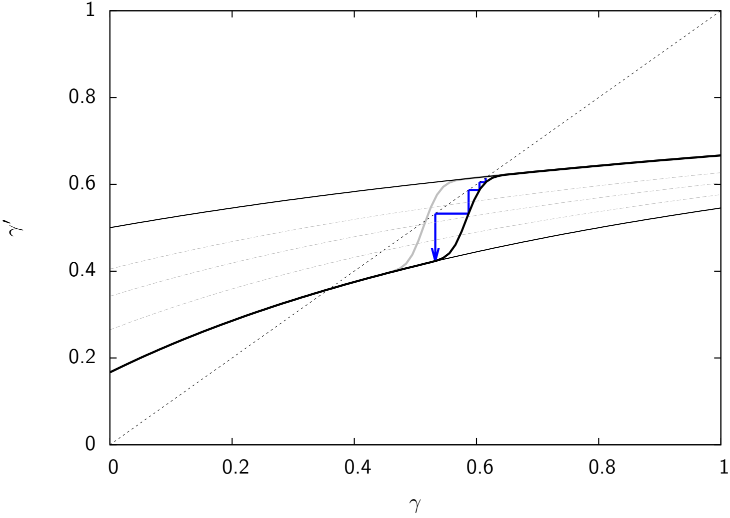
Uncertainty Traps: Falling in the Trap

Impact of a **medium**-sized negative shock to μ



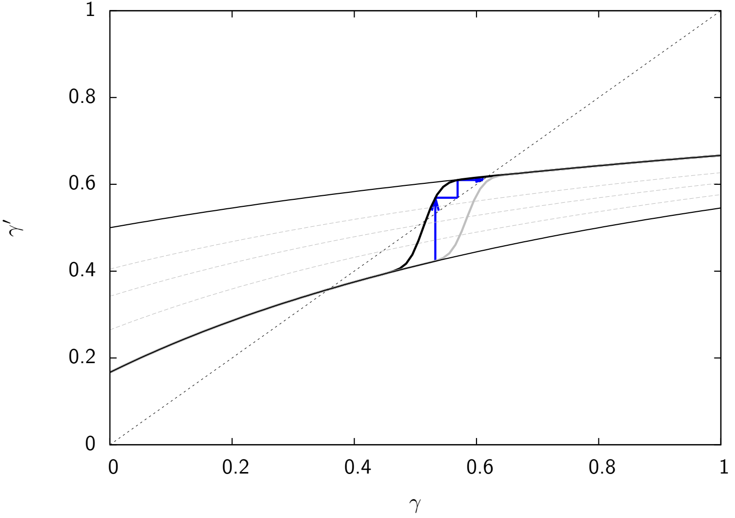
Uncertainty Traps: Falling in the Trap

Impact of a **medium**-sized negative shock to μ

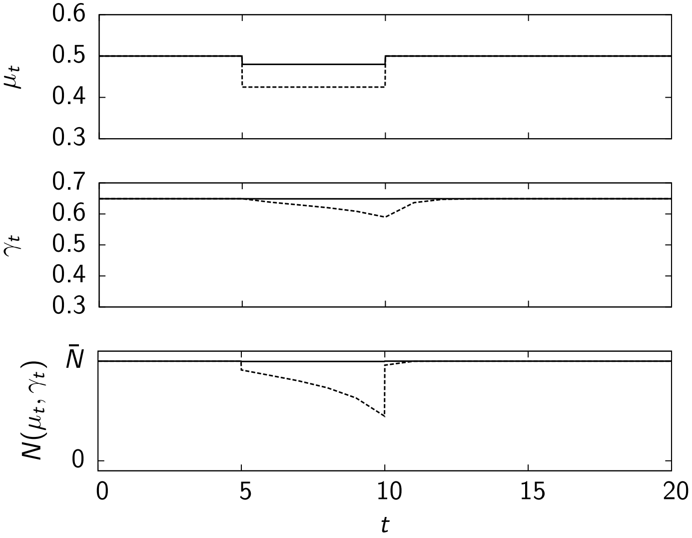


Uncertainty Traps: Falling in the Trap

Impact of a **medium**-sized negative shock to μ

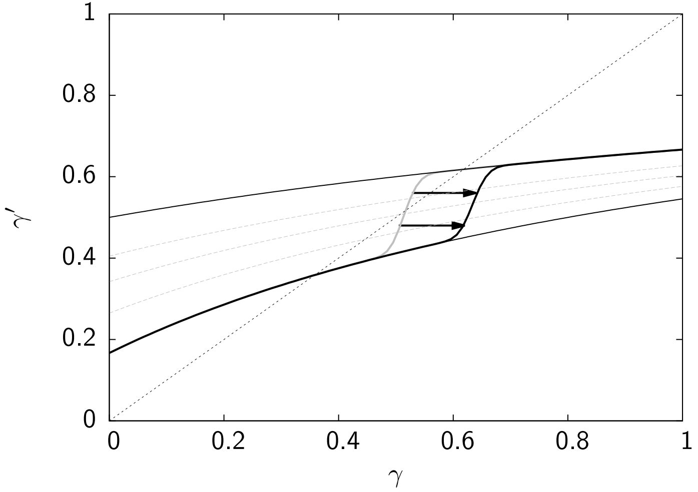


Uncertainty Traps: Falling in the Trap



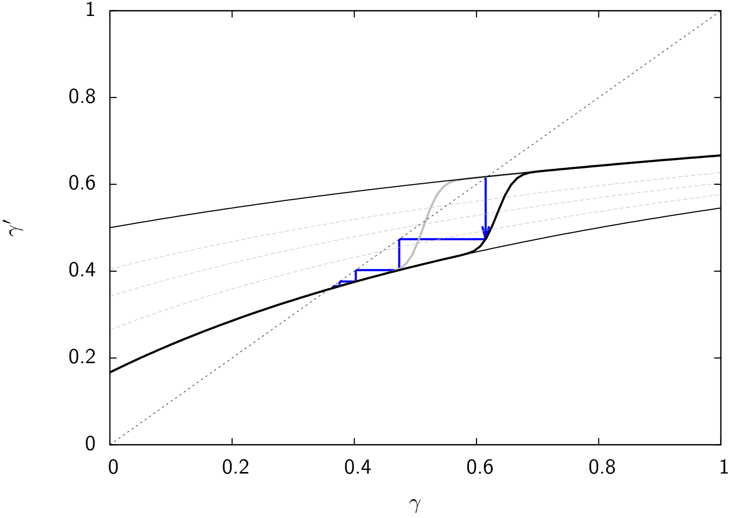
Uncertainty Traps: Falling in the Trap

Impact of a **large** negative shock to μ



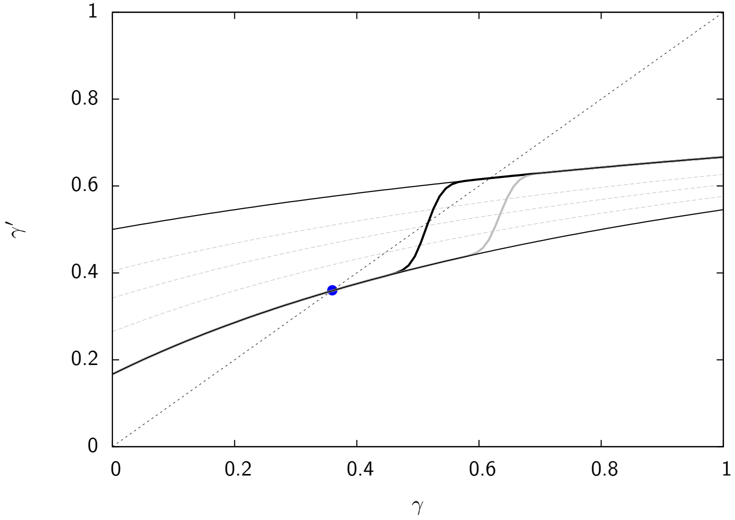
Uncertainty Traps: Falling in the Trap

Impact of a **large** negative shock to μ

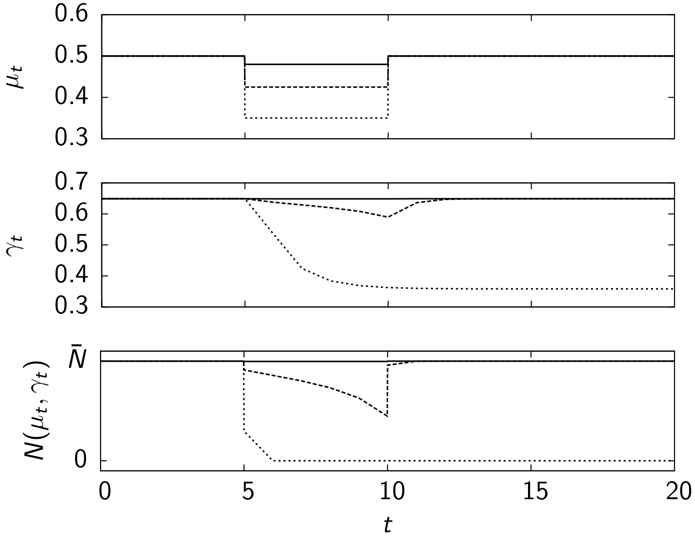


Uncertainty Traps: Falling in the Trap

Impact of a **large** negative shock to μ



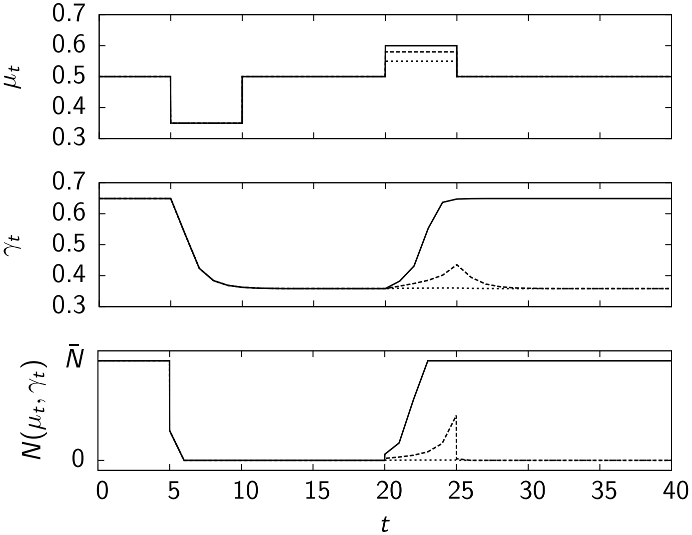
Uncertainty Traps: Falling in the Trap



Uncertainty Traps: Escaping the Trap

- We now start after a full shift of the economy towards the low regime
- How can the economy escape the trap?

Uncertainty Traps: Escaping the Trap



Uncertainty Traps

- The economy displays strong non-linearities:
  - for small fluctuations, uncertainty does not matter much,
  - only large or prolonged declines in productivity (or signals) lead to self-reinforcing uncertainty events: uncertainty traps
- In such events, the economy may remain in a depressed state even after mean beliefs about the fundamental recover (μ)
  - Jobless recoveries, high persistence in aggregate variables
- The economy can remain in such a trap until a large positive shock hits the economy

Welfare Implications

- The economy is inefficient because of an informational externality
  - Firms do not internalize the effect of their investments on public information

Proposition

The following results hold:
1) The competitive equilibrium is inefficient. The socially efficient allocation can be implemented with positive investment subsidies $\tau(\mu, \gamma)$*;*
2) In turn, uncertainty traps may still exist in the efficient allocation.

Extended Model

- Robustness:
  - Neoclassical production functions with capital and labor
  - Mean-reverting process for θ
  - Long-lived firms that accumulate capital over time
  - Firms receive investment opportunities stochastically

Extended Model - Summary

- Representative risk neutral household owns firms and supplies labor
- CRS production technology in capital and labor:

$$(A + Y)\, k_n^{\alpha} l_n^{1-\alpha}$$

  with $Y = \theta + \varepsilon^{Y}$ and $\theta' = \rho_{\theta}\theta + \varepsilon^{\theta}$
- Firms accumulate capital over time: $k_n' = (1 - \delta + i)\, k_n$
- Convex cost of investment: $c(i) \cdot k_n$
- Fixed cost of investment: $f \cdot k_n$
- Stochastic arrival of investment opportunity with probability $\overline{q}$
  - Denote Q the total stock of firms with an opportunity
- Economy aggregates easily thanks to linearity in k_n (Hayashi, 1982)

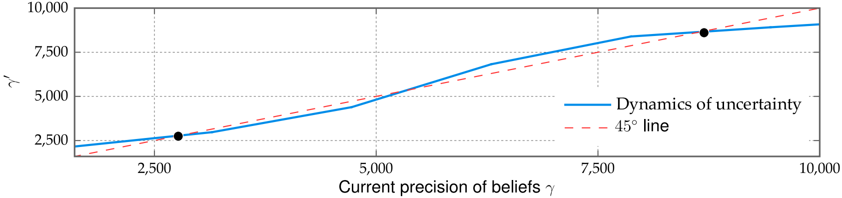
▸ Timing ▸ Information ▸ Planner

Numerical Example - Parametrization

Parameter	Value
Time period	Month
Total factor productivity	$A = 1$
Discount factor	$\beta = (0.95)^{1/12}$
Depreciation rate	$\delta = 1 - (0.9)^{1/12}$
Share of capital in production	$\alpha = 0.4$
Probability of receiving an investment opportunity	$\overline{q} = 0.2$
Cost of investment	$f = 0.1$
Variable cost of investment $c(i) = i + \phi i^2$	$\phi = 10$
Persistence of fundamental	$\rho = 0.99$
Precision of ergodic distribution of fundamental	$\gamma_\theta = 400$
Precision of public signal	$\gamma_y = \underline{100}, 1000, 5000$
Precision of aggregated private signals when $N = 1$	$\gamma_x = 500, \underline{1500}, 5000$

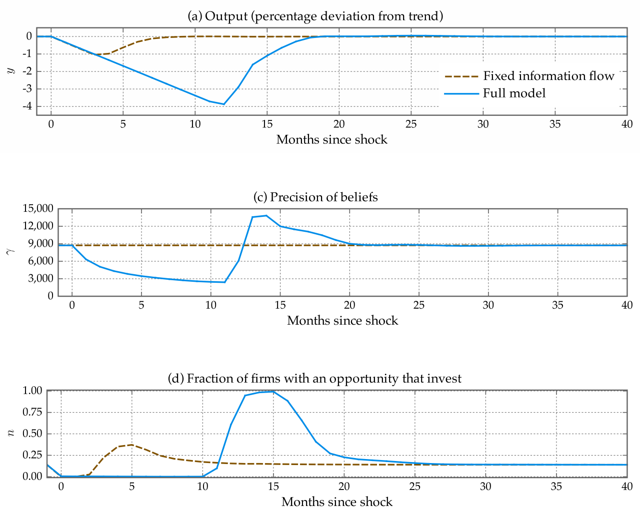
Table: Parameters values for the numerical simulations

Numerical Example: Dynamics of Uncertainty

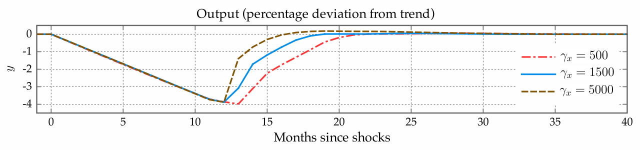


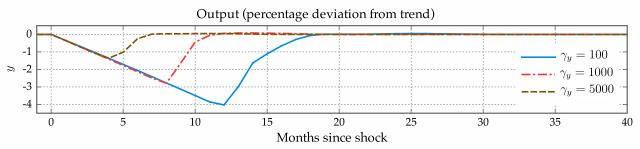
- Multiple stationary points in the dynamics of γ still obtain
  - But other state variable evolve in the background: K and Q
  - In a trap, as K reaches a low, firms start investing
- The economy is unlikely to remain in a trap forever, but we may still have persistence

Numerical Example: Negative 5% shock to μ

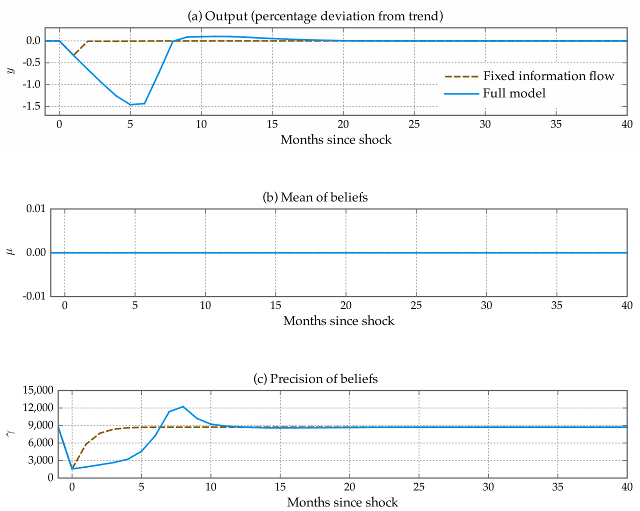


Numerical Example: Sensitivity





Numerical Example: Negative 50% shock to γ



Numerical Example

- Results:
  - Endogenous uncertainty substantially increase the persistence of recessions vs. constant uncertainty in an RBC model
  - The additional persistence is large for a wide range of values for γ_x, it is however important that γ_y is not too high for uncertainty to matter
- Key challenge:
  - How to identify/measure the information parameters in the data for full quantitative evaluation

▸ Evidence

Conclusion

- We have built a theoretical model in which uncertainty fluctuates endogenously
- The complementarity between economic activity and information leads to uncertainty traps
- Uncertainty traps are robust to more general settings
  - Full quantitative evaluation using firm-level data on investment and expectations
  - Uncertainty on industry-level productivity or aggregate TFP growth
- Interesting extensions:
  - Monopolistic competition: people not only care about the fundamental but also about the beliefs of others (higher-order beliefs)
  - Financial frictions: amplification through risk premium

Equilibrium Characterization

Proposition

If $\beta e^{\frac{a^2}{2\gamma_\theta}} < 1$ and F is continuous, twice-differentiable with bounded first and second derivatives, for γ_x small,

1) The equilibrium exists and is unique;
2) The investment decision of firms is characterized by the cutoff $f_c(\mu, \gamma)$ such that firms invest iff $f \leq f_c(\mu, \gamma)$;
3) f_c is a strictly increasing function of μ and γ.

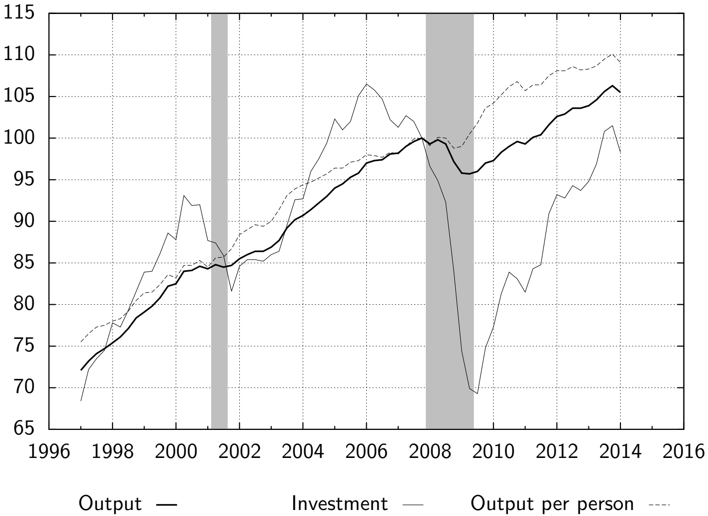
Return

Limit $N \to \infty$

- If γ_x was constant as we take the limit, a law of large number would apply and θ would be known
- To prevent agents from learning too much, we assume $\gamma_x\left(\bar{N}\right) = \gamma_x / \bar{N}$. Therefore the precision of the aggregate signal X stays constant at
$$N\gamma_x(\bar{N}) = n\gamma_x$$
where
$$n = \frac{N}{\bar{N}}$$
is the fraction of firms investing.
- Under this assumption, the updating rules for information are the same as with finite N

Return

2007-2009 Recession



Return

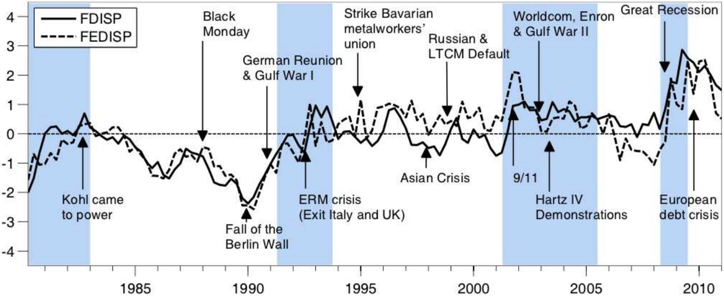
Suggestive evidence

- Our theory predicts that deep recessions are accompanied by
  - High subjective uncertainty ▸ Germany ▸ Italy ▸ UK ▸ US
  - Increased firm inactivity ▸ Literature ▸ Compustat
- We provide purely suggestive evidence
  - Data is extremely limited and difficult to interpret
  - Causality is hard to identify ▸ VAR

◂ Roadmap ◂ Numerical example

Some suggestive evidence: Dispersion of Beliefs

- Bachmann, Elstner and Sims (2012):
  - Survey of 5,000 German businesses (IFO-BCS)
  - Compute variance of ex-post forecast error about general economic conditions (FEDISP) and a dispersion of beliefs (FDISP)



◂ Return

Some suggestive evidence: Italy

- Bond, Rodano and Serrano-Velarde (2013):
  - Survey of Industrial and Service Firms (Bank of Italy)
  - All firms with 20 or more employees in industry or services

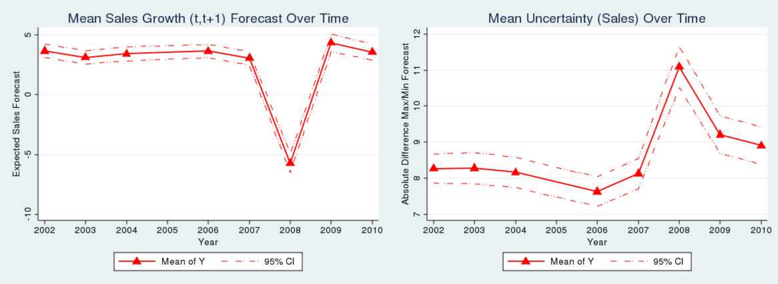


Figure: Mean and variance of expected sales

Return

Some suggestive evidence: CBI

- CBI Industrial Trend Survey:
  - Monthly survey of CEOs across 38 manufacturing sectors
  - Factors likely to limit capital investment in the next 12 months

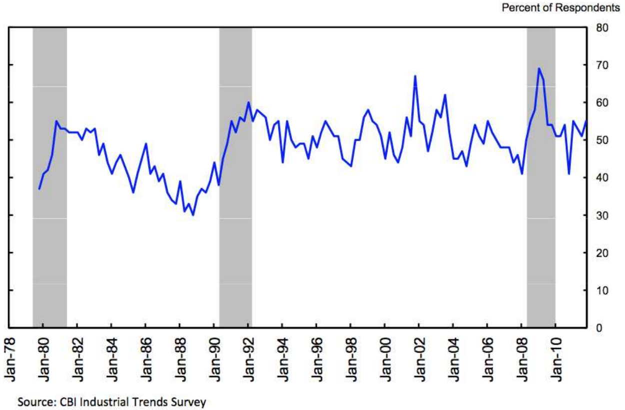
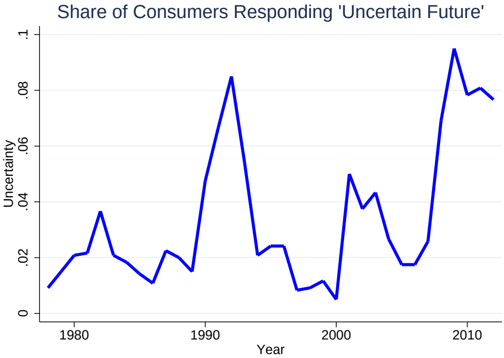


Figure: Fraction of responses 'uncertain demand' (Leduc and Liu, 2013)

Return

Some suggestive evidence: Uncertainty over the Business Cycle

- National Federation of Independent Businesses 2012 Survey ranks the most severe problems facing small business owners:
  - 40% of respondents ranked economic uncertainty as the main problem that they faced in 2012
- Michigan Survey of Consumers: main reason why it is not a good time to buy a car (% of households)



Return

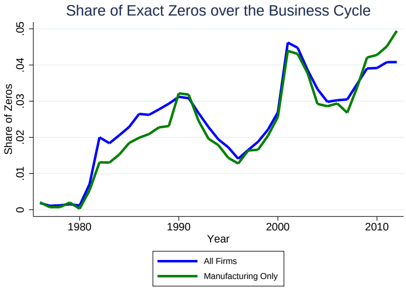
Some suggestive evidence: Firm Inactivity over the Business Cycle

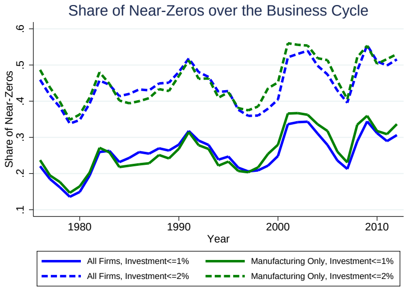
- Prevalence of inactivity during recessions
  - Cooper and Haltiwanger (2006): 8% of firms in the US have near-zero investment ($< 1\%$ in absolute value) between 1972 and 1988
  - Gourio and Kashyap (2007): correlation of -0.94 between aggregate investment and share of investment zeros in the US between 1975 and 2000
- Carlsson (2007):
  - Estimates neoclassical model with irreversible capital using US firm-level data
  - Uncertainty (volatility in TFP and factor prices) has negative impact on capital accumulation in short and long run
  - Large SR effect, moderate LR: 1 SD increase in uncertainty leads to a drop of 16% of investment in SR, 2% if permanent

Return

Some suggestive evidence: Firm Inactivity and Uncertainty

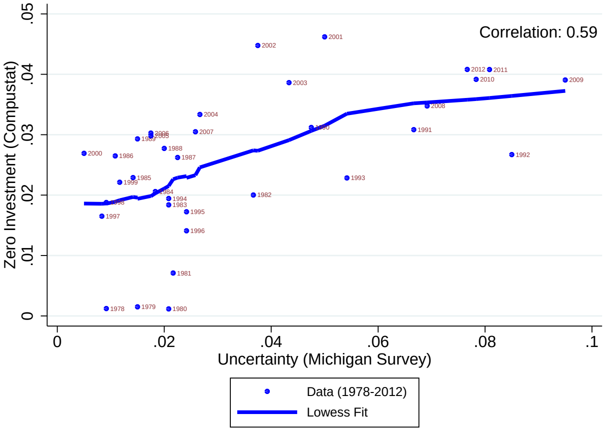
- Evidence from Compustat





Some suggestive evidence: Firm Inactivity and Uncertainty

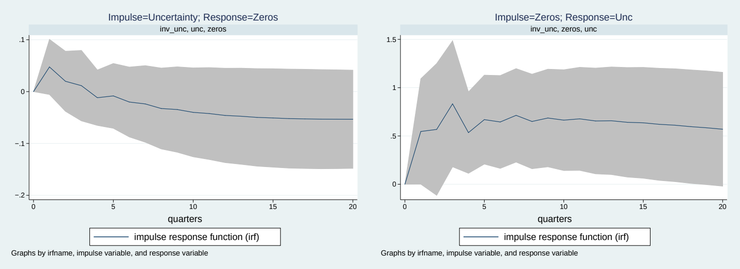
- Correlation firm inactivity (Compustat) and uncertainty (Michigan Survey)



Return

VAR Evidence

- Simple bivariate VAR with investment zeros and uncertainty
  - No contemporaneous effect of 0s on uncertainty



Return

Timing

1. At the beginning, all firms share the same prior distribution on θ

$$\theta|\mathcal{I} \sim \mathcal{N}\left(\mu, \gamma^{-1}\right)$$

2. Firms without investment opportunities receive one with probability $\overline{q}$
3. Firms with an investment opportunity decide whether or not to invest
4. Investing firms receive a private signal $x_n = \theta + \varepsilon_n^x$ and choose labor l_n
5. The aggregate shock Y is realized, individual actions are observed
6. Production takes place, markets clear
7. Agents update their beliefs

Return

Information

- The structure of information is the same as before
  - Assume, in addition, that each firm knows its individual state and the productivities and capital stocks of others.
- Revealing equilibria:
  - individual private signals x_n are revealed through firms' hiring decisions
  - summarize by public signal X with precision $N\gamma_x$
- Belief dynamics

$$\mu' = \rho_\theta \frac{\gamma\mu + \gamma_y Y + \gamma_x \left(\int q_j \chi_j k_j dj\right) X}{\gamma + \gamma_y + \gamma_x \int q_j \chi_j k_j dj} = \rho_\theta \frac{\gamma\mu + \gamma_y Y + nQ\gamma_x X}{\gamma + \gamma_y + nQ\gamma_x}$$

$$\gamma' = \left(\frac{\rho_\theta^2}{\gamma + \gamma_y + \gamma_x \int q_j \chi_j k_j dj} + \frac{1-\rho_\theta^2}{\gamma_\theta}\right)^{-1} = \left(\frac{\rho_\theta^2}{\gamma + \gamma_y + nQ\gamma_x} + \frac{1-\rho_\theta^2}{\gamma_\theta}\right)^{-1}$$

Return

Extended Model - Planner

- The planning problem in this economy is

$$V\left(\mu,\gamma,\{k_j,q_j\}\right)=\max_{\{i_j,k_j,l_j\}}\mathbb{E}\left\{U\left(\left(A+Y\right)\int_0^1 k_j^{\alpha}l_j^{1-\alpha}dj\right.\right.$$
$$\left.\left.-\int_0^1\left(f+c\left(i_j\right)\right)k_jq_j\chi_jd_j\right)+\beta V\left(\mu',\gamma',\{k_j',q_j'\}\right)\right\}$$

subject to

$$1=\int_0^1 l_j dj$$
$$k_j'=q_j\chi_jk_j\left(1-\delta+i_j\right)+\left(1-q_j\chi_j\right)k_j\left(1-\delta\right)$$
$$q_j'=q_j\left(1-\chi_j\right)+\left(1-q_j+q_j\chi_j\right)\begin{cases}0 & \text{w.p. } 1-\overline{q}\\ 1 & \text{w.p. } \overline{q}\end{cases}$$

and laws of motion for information.

Extended Model - Planner

- The planning problem aggregates into

$$V(\mu, \gamma, K, Q) = \max_{i,n \in [0,1]} \mathbb{E}\{U((A+\mu)K^{\alpha} - nQ(f + c(i))) \\ + \beta V(\mu', \gamma', K', Q')\}$$

subject to

$$K' = (1-\delta)K + inQ$$
$$Q' = (1-\delta)(1-\overline{q})(1-n)Q + (1-\delta)\overline{q}K + \overline{q}inQ$$

and laws of motion for information, where $K = \int k_j dj$ and $Q = \int k_j q_j dj$.

Return