Interbanking Networks: Controlled Cascade Failures Through Taxation*

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1 Introduction

Since the collapse of Lehman Brothers, much attention has been given to the systemic nature of the financial system. The degree of connections between various portfolios, leading in part to high leveraging ratios, shown the importance to understand not only the balance sheet of prospective investments, but also the balance sheets of porfolios directly or indirectly connected to them. In short, the crisis has pointed a need for a better understanding of the interdependance of the banking system, or stated otherwise, the interbanking network.

This paper further investigates the understanding of interbanking connections and how its undesirable effects can be mitigated. I first use a simple model of network formation in a static equilibrium setup to discuss the formation and importance of risk externalities on the network. I provide a characterization of the severity of the externality in terms of heterogeneity in the banking system. I show that only when there is diversity in banks' abilities to extract returns from assets the network externality never vanishes. If banks are on the opposite homogenous, they contribute to the formation of a regular network, which in equilibrium eliminates the externality. In particular, increasing the ratio of "attractive banks" decreases the externality, regardless of the market size.

When there is heterogeneity, I move on to the correction of the externality through the use of tax instruments. I show that the first best can be reached if the number of tax instruments is as diverse as the number of different connections (or equilibrium shares).

*Provisory catchy title.

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The model is then slightly modified to include dynamic effects such as cascades of failures, e.g. when the failure of one bank lead to the failure of some other banks through a non-conservative law[1]. Much of the results pertaining to taxation extends to the dynamic framework, and the design of the tax system is so that in equilibrium, the loss remains minimal. The paper thus provides some guidance on the design of incentives that leads to optimal network formation.

In the dynamic setup, the model is designed to avoid a state space cursed with dimensionality. If a generic shock distribution is used on fundamental assets, the state space of the model becomes too large. By separating the process of asset selection from the magnitude of the shocks (much as in Blume, Easly, Kleinberg, Kleinberg & Tadros, 2011) the state space can be managed in polynomial time and thus applied to large connections. This use of fundamental shocks and a low state space also makes the model appealing for practical purposes: it avoids the multiplicity of equilibria and can be applied to networks with a large number of connections.

1.1 How This Paper Relates to the Literature (draft)

This paper is connected to three streams of literature. The first one bears to cascade of failures as in (Elliot, Golub & Jackson) or (Blume, Easly, Kleinberg, Kleinberg & Tadros). The first paper studies how various exogenous banking networks lead to cascades of failures while the second provides a endogenous network formation with a simple probabilistic rule for cascades.

The second stream of deals with congestion on networks (as in trafic networks), and how tolls can be used to correct for overcrowding as in (Kelly), (Cole, Dodis & Roughgarden) or (Karakostas & Kolliopoulos). These papers study how, given a linear reaction function of traffic users, tolls can correct an inefficient Nash equilibrium (or Wardrop equilibrium, in the computer science literature). Some discussions on the complexity of finding solutions are also found.

There is finally a stream of literature that does not pertain to the network framework, but discuss the relationship between taxation and market externalities (an example devoted to the banking system is given in Bierbauer, JET, forthcoming). The focus of the papers is then on efficiency of the market, the incentives provided to economic agents and the optimal method to collect governmental revenues.

[1] Some use a discontinuity in value, or a “sharp drop” to represent such non-conservative law.

Depending on the perspective, this paper contributes to the literature in various ways. It provides a fully endogenous model of interbanking network with a single equilibrium. It shows that if a government uses tax instruments for the sole role of redistributing incentives across investments (e.g. a zero revenue tax), it can correct the externality and provide a first best outcome. It also shows the relationship between the severity of the externality and the size of the market, and thus, the relationship between taxation and the size of the market. In a dynamic setup, it also avoids the multiplicity of equilibria in values (as explicited by Elliot, Golub & Jacskon) by letting nature weight the likelihood of each failure (through a probability distribution). By doing so, each agent is able to rate each outcome and thus provide a unique value *ex-ante*.

I begin in section 2 with a brief overview of the notation and some static analysis of the model. I then move on to results in optimal taxation in section 3 and characterize the optimal decentralized tax system. I also provide a discussion on a second-best result where the government must tax assets in a similar way for each investor. In 4, I slightly change the setup to discuss the results in the context of a dynamic model.

2 Model & Notation

2.1 Notation

The economy is composed of portfolios of assets, which I call **banks** for brevity, and of **fundamental assets**. Banks are indexed by $b, b' \in B$ and fundamental assets $f \in F$. I will use the index $a \in A \equiv B \cup F$ to cover all assets in the economy. The number of elements in these sets are denoted respectively by $|B|, |F|, |A|$ and likewise for other sets. Each bank has an initial ownership of some fundamental assets and I denote this set $f(b) \equiv \{f' \in F | b \text{ initially owes } f'\}$.

The **share** that bank b owns in asset a is denoted $x_{ba} \in [0,1]$. It is useful to decompose the global matrix of shares X in various fashions. I denote $X_B \in [0,1]^{|B|\times|B|}$ the square **matrix of interbank investments** where each element $x_{bb'}$ is the share of investments of one bank in another bank, and $X_F \in [0,1]^{|B|\times|F|}$ the **matrix of fundamental investments**, where each component x_{bf} is a share of ownership in fundamental asset f. For the purpose of discussion, investments decisions needs to be separated between the endogenous variables of bank $b, \mathbf{x}_b \in \mathbb{R}^{|A|}$, and any given ownership decision made by other banks $X_{!b} \in [0,1]^{|A|\times(|A|-1)}$. Finally, for the analysis of returns on fundamental investments, the matrix $X_{Fr} \in [0,1]^{|B|\times|B||F|}$, defined below, becomes handy:

$$X_{Fr} \equiv \begin{bmatrix} x_{b_1 f_1} & \dots & x_{b_1 f_F} & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & \dots & 0 & x_{b_2 f_1} & \dots & x_{b_2 f_F} & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 & \dots & x_{b_B f_1} & \dots & x_{b_B f_F} \end{bmatrix} \tag{1}$$

Banks are heterogenous in their ability to extract profits from other assets. So, **average returns** on assets are heterogenous and denoted $\tilde{r}_{ba} \in \mathbb{R}^+$. Similarly, **net average returns** are denoted $r_{ba} \equiv (1 - \tau_{ba})\tilde{r}_{ba}$, where τ_{ba} is the tax rate on the transaction x_{ba}. In general, bold symbols refer to a vector of elements. For example, $\mathbf{r}_b \in \mathbb{R}^{|A|}, \mathbf{r}_{BA} \in \mathbb{R}^{|B||A|}$ and $\boldsymbol{\sigma}^2 \in \mathbb{R}^{|A|}$ respectively refer to the vector of net returns of bank b, the vector that stacks all returns in one vector and the vector of variances on assets. As assets are random variables, it is useful to denote $\boldsymbol{\epsilon}_B \in \mathbb{R}^{|B|}$ the **vector of random shocks** to each of the banks and similarly, $\boldsymbol{\epsilon}_f \in \mathbb{R}^{|F|}$ denotes the vector of random shocks on fundamental assets.

There is finally some standard and miscellaneous notation used throughout the paper: $\mathbf{p} \in \mathbb{R}^{|A|}$ refers to prices, $\boldsymbol{\omega} \in \mathbb{R}^{|B|}$ are pareto weights, endogenous variables appended with a star (for instance, x^*_{ba}) refer to chosen allocation quantity while when appended with "opt" (x^{opt}_{ba}), it refers to the efficient quantity.

2.2 Model

2.2.1 Assets, Banks and Shock Spillovers

The economy is composed of two types of assets: banks and fundamental assets. Each asset in the economy is sold in shares than can be bought or sold. The sum of these shares is equal to one. A fundamental asset f follows some distribution with mean zero and a given variance σ_f^2 and each are uncorrelated. Banks are portfolios of assets and they can invest in fundamental assets or in other banks by buying shares. Each bank is owned by a bank owner whose sole investment vehicle is its own bank.

Banks are heterogenous in their ability to extract profits from fundamental assets and so the average returns on assets $\tilde{r}_{bf} > 0$ are different from one bank to another. Asset f is thus seen as a distribution with mean r_{bf} and variance σ_f^2 from b's point of view. This heterogeneity in the ability to extract profits influences the decision to invest in a particular asset and thus, will influence the network topology X_B^*, X_F^* in equilibrium. So although all returns r_{bf} are positive for banks, the ownership structure varies.

From the perspective of bank b, a fundamental asset f produces an output ϵ_f according to its distribution. So if bank b holds x_{bf} shares in asset f, its return is given by $(\tilde{r}_{bf} + \epsilon_f)x_{bf}$ and thus inherits a shock $x_{bf}\epsilon_f$. This means that bank b' inherits a shock $x_{b'b}\epsilon_b$ from bank b, where ϵ_b is the resulting shock of bank b's portfolio. The shock that bank b' receives thus depends on the weighted sum of shocks in other assets. Some of these assets, banks, also depend on investments in other banks and therefore, a given return in a bank depends on the global network of ownership:

$$\tilde{r}_b + \epsilon_b = \sum_{a \in A} x_{ba}(\tilde{r}_{ba} + \epsilon_a), \ \forall a. \tag{2}$$

A separation of shocks and returns allows for an equivalent in matrix notation:

$$\boldsymbol{\epsilon}_B = X_B \boldsymbol{\epsilon}_B + X_F \boldsymbol{\epsilon}_F, \tag{3}$$

$$\tilde{\boldsymbol{r}}_B = X_B \tilde{\boldsymbol{r}}_B + X_{Fr} \tilde{\boldsymbol{r}}_{BF}. \tag{4}$$

The solution to those two equations can be found through recursion and assuming that X_B has eigenvalues smaller than one have a finite solution:

$$\boldsymbol{\epsilon}_B = \underbrace{(I - X_B)^{-1} X_F \boldsymbol{\epsilon}_F}_{\equiv \epsilon(X_B, X_F)}, \tag{5}$$

$$\tilde{\boldsymbol{r}}_B = \underbrace{(I - X_B)^{-1} X_{Fr} \tilde{\boldsymbol{r}}_{BF}}_{\equiv r(X_B, X_F)}. \tag{6}$$

As eigenvalues of X_B need to be smaller than one, some discussion is in order about their interpretation: they represent the geometric amplification of a linear combination of the cycles on the inter banking network. Imagine a network with a single cycle of two banks, indexed one and two. Aside from fundamental assets, each bank can only invest in each other, forming a cycle x_{12}, x_{21}. Then, the sole eigenvalue of the inter banking topology is given by $\sqrt{x_{12}x_{21}}$, which gives the (geometric) average level of amplification of a shock by passing through one of the two branches of the cycle. If this number is greater than one, then fundamental shocks passing through this cycle are amplified up to infinity. If the payoff of banks depends negatively on the variance of shocks, the disutility of an infinite variance is greater than the utility of infinite returns. So since shocks on the cycle includes those by the two banks themselves, it cannot be consistent with any positive investment (and thus, any positive eigenvalue). It cannot therefore be part of an equilibrium. As long as payoffs depends negatively on the variance, this line of reasoning holds for any cycle. As I model payoffs below with such property, I assume throughout that no eigenvalues on X_B are larger than one in equilibrium.

The recursive solution to 3 can be presented as an infinite geometric sum on X_B:

$$(I - X_B)^{-1} = \sum_{i=0}^{\infty} (X_B)^i, \tag{7}$$

which shows also that any interbanking matrix X_B acts as an amplifier on fundamental shocks. Let $V \Lambda V^{-1} = X_B$ be the eigenvector/eigenvalue decomposition of X_B, the infinite sum shows that:

$$\boldsymbol{\epsilon}_B = V \left(\sum_{i=0}^{\infty} \Lambda^i \right) V^{-1} X_F \boldsymbol{\epsilon}_F. \tag{8}$$

As the matrix of eigenvalues Λ is a diagonal matrix, the infinite sum is nothing but the geometric sum of the eigenvalues:

$$\left(\sum_{i=0}^{\infty} \Lambda^i \right) = \begin{bmatrix} \frac{1}{1-\lambda_1} & 0 & \dots & 0 \\ 0 & \frac{1}{1-\lambda_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{1-\lambda_{|B|}} \end{bmatrix}, \tag{9}$$

where λ_i is any of the $|B|$ eigenvalues of the interbanking investment matrix. Because each eigenvalue is smaller than one, any element $\frac{1}{1-\lambda_i}$ is greater than one. Using the same interpretation of geometric averages on cycles, this shows that the interbanking network amplifies shocks, each banking cycle contributing to an increase in shocks by a factor of $\frac{1}{1-\lambda_i}$. These amplification cycles are then redistributed amongst banks according to the eigenvectors contained in V (which could be called the redistributive effect of X_B).

Notice that 5 defines the shocks $\boldsymbol{\epsilon}_B$ as a function of X_B, X_F and the fundamental shocks $\boldsymbol{\epsilon}_F$. The next proposition has the most important result that $\boldsymbol{\epsilon}_B$ is convex in any element of X_B:

Proposition 1. *Let $\boldsymbol{\epsilon}_B(X_B, X_F)$ be the function of X_B, X_F defined in 5 and denote any component $\epsilon_b(X_B, X_f), b \in B$. Then, the following statements are true:*

1. *$\epsilon_b(X_B, X_F)$ is convex for any $b' \in B$:*

$$\frac{\partial \epsilon_b(X_B, X_F)}{\partial x_{bb'}} \geq 0 \ \forall b' \in B, \qquad \frac{\partial \epsilon_b(X_B, X_F)}{\partial x_{bb'} \partial x_{bk}} \geq 0 \ \forall b', k \in B. \tag{10}$$

2. *If there exists a path of ownership between bank b and asset f, then ϵ_b depends on ϵ_f. Furthermore, for any bank b' on a path between bank b and asset f, $\epsilon_b(X_B, X_F)$ is linear in $x_{b'f}$.*

Proof. See Appendix A.1 □

A direct corollary of these results is that the variance of each bank is a convex function of X_B and X_F, since $\sigma_b^2 = \mathbb{E}\epsilon_b(X_B, X_F)^2$. This means in particular that the variance of bank b depends on the investment decisions made by bank b'. As banks b' are only interested in maximizing their own portfolio, this creates an externality on risk and return.

This externality can be understood through the global variance-covariance: For some given equilibrium shares, the matrix can be expressed as:

$$\mathbb{E}\boldsymbol{\epsilon}_b = (I - X_b^*)^{-1} X_f^* \mathbb{E}\boldsymbol{\epsilon}_F \boldsymbol{\epsilon}_F' X_f^{*'} (I - X_b^*)^{-1}. \tag{11}$$

In traditionnal portfolio theory, this matrix is considered fixed and independant of banks' decisions (leading to a quadratic problem). In the current setup, the variance covariance matrix and the returns on assets becomes endogenous.

A direct implication of having the eigenvalues of X_B smaller than one is that some banks do not fully sell their ownership to other banks. I adopt the convenient explanation in Elliot, Golub & Jacskon by saying that this is the value held by the bank owners, or owners external to the network. As they very well explain, these external owners face no amplification on their ownership (or on their book value), as the amplification effect of the inter banking network is exactly compensated by the share of external ownership. In other words, if the diagonal matrix of external ownership is labelled $\hat{X}_B$[2], the product of $\hat{X}_B(I - X_B)^{-1}$ yields a matrix whose eigenvalues are all equal to one. As this papers focuses on the ability of banks to sustain downward shocks, I focus however on the portfolio of banks themselves, which face amplification.

[2]Such matrix is diagonal and has the property that the sum of the elements of columns i in both $\hat{X}_B$ and X_B is equal to one, for any given column i.

2.2.2 A Simple Example

I provide in figure 1(a) an example with two banks and two assets. For a given level of investment X_B and X_F the solution to equation 5 is given by:

$$\epsilon_2(X_B, X_F) = \frac{x_{23}x_{34} + x_{24}(1 - x_{33})}{(1 - x_{33})(1 - x_{22}) - x_{23}x_{32}}\epsilon_4 + \frac{x_{23}x_{31} + x_{21}(1 - x_{33})}{(1 - x_{33})(1 - x_{22}) - x_{23}x_{32}}\epsilon_1, \tag{12}$$

$$\epsilon_3(X_B, X_F) = \frac{x_{32}x_{24} + x_{34}(1 - x_{33})}{(1 - x_{33})(1 - x_{22}) - x_{23}x_{32}}\epsilon_4 + \frac{x_{32}x_{21} + x_{31}(1 - x_{33})}{(1 - x_{33})(1 - x_{22}) - x_{23}x_{32}}\epsilon_1. \tag{13}$$

Because shocks on fundamental assets are independent, the variance on banks one and two are given by:

$$\sigma_2^2 = \left(\frac{x_{23}x_{34} + x_{24}(1 - x_{33})}{(1 - x_{33})(1 - x_{22}) - x_{23}x_{32}}\right)^2 \sigma_4^2 + \left(\frac{x_{23}x_{31} + x_{21}(1 - x_{33})}{(1 - x_{33})(1 - x_{22}) - x_{23}x_{32}}\right)^2 \sigma_1^2, \tag{14}$$

$$\sigma_3^2 = \left(\frac{x_{32}x_{24} + x_{34}(1 - x_{33})}{(1 - x_{33})(1 - x_{22}) - x_{23}x_{32}}\right)^2 \sigma_4^2 + \left(\frac{x_{32}x_{21} + x_{31}(1 - x_{33})}{(1 - x_{33})(1 - x_{22}) - x_{23}x_{32}}\right)^2 \sigma_1^2. \tag{15}$$

[Figure 1 about here]

The returns on banks can also be found in a similar fashion:

$$r_2(X_B, X_F) = \frac{x_{23}x_{34}r_{34} + x_{24}(1 - x_{33})r_{24}}{(1 - x_{33})(1 - x_{22}) - x_{23}x_{32}} + \frac{x_{23}x_{31}r_{31} + x_{21}(1 - x_{33})r_{21}}{(1 - x_{33})(1 - x_{22}) - x_{23}x_{32}}, \tag{16}$$

$$r_3(X_B, X_F) = \frac{x_{32}x_{24}r_{24} + x_{34}(1 - x_{33})r_{34}}{(1 - x_{33})(1 - x_{22}) - x_{23}x_{32}} + \frac{x_{32}x_{21}r_{21} + x_{31}(1 - x_{33})r_{31}}{(1 - x_{33})(1 - x_{22}) - x_{23}x_{32}}. \tag{17}$$

2.2.3 Banks' Decisions and Market Equilibrium

I restrict my attention to a reduced form of portfolio valuation measuring the trade-off between the variance and returns. I use the following measure of payoff:

$$u(\mathbf{x}_b, \boldsymbol{r}_b, \boldsymbol{\sigma}^2, X_{!b}) = \sum_a r_{ba}x_{ba} - h\left(\sum_a \sigma_a^2 \frac{x_{ba}^2}{2}\right) \tag{18}$$

with h being increasing and strictly convex. This functional form assumes that banks measures the variance and the return of their portfolio equally for each of their shareholder. It also assumes a form of anonymity as they are indifferent in the source of variance or average returns. The biggest assumption is separability between risk and return and is made to keep the theoretical analysis tractable. Banks make their portfolio decision *ex-ante* and are guided by:

$$u^*(\mathbf{p}, \mathbf{r}_b, \boldsymbol{\sigma}^2, X_{!b}) \equiv \max_{\mathbf{x}_b} u(\mathbf{x}_b, \boldsymbol{r}_b, \boldsymbol{\sigma}^2)$$

$$\text{s.t.} \sum_{f' \in f(b)} p_{f'} + p_b = \sum_{a \in A} p_a x_{ba},$$

$$\sigma_{b'}^2 = \mathbb{E}\epsilon_{b'}(X_B, X_F)^2 \; \forall b', \qquad r_{bb'} = r_{b'}(X_B, X_F) \; \forall b' \tag{19}$$

Because u is at least quadratic in the vector $(X_F'(1 - X_B)^{-1}\boldsymbol{x}_b, \boldsymbol{x}_f)$ and there is a unique relationship between $(X_F'(1 - X_B)^{-1}\boldsymbol{x}_b$ and $\boldsymbol{x}_b$, the payoff of banks remains concave in $\boldsymbol{x}_b, \boldsymbol{x}_f$ and the first order condition of bank b with respect to asset a is given by:

$$0 = r_{ba} - h'(\cdot)\sigma_a^2 x_{ba} + \sum_{b' \in B} \frac{\partial r_{b'}}{\partial x_{ba}} x_{bb'} - h'(\cdot) \sum_{b' \in B} \frac{\partial \sigma_{b'}^2}{\partial x_{ba}} \frac{x_{bb'}^2}{2} - Lp_a \; \forall b \in B, a \in A \tag{20}$$

The first two terms on the right hand side measure the standard risk versus return dimension of the portfolio. The two following terms captures the network effect of bank b on its own payoff. An increase in the share of asset a induces an increase in the return of all other banks, (assuming they have a non-zero investment in b) which increases the payoff of bank b through its own investments in other banks. Similarly, an increase in the shareholding of a increases the variance of all other connected banks, which decreases in return the payoff. As the variance increases faster than the return, these two terms are negative altogether.

The following proposition summarizes the optimal portfolio $\mathbf{x}_b^*(\mathbf{p}, \mathbf{r}_b, \boldsymbol{\sigma}^2, X_{!b})$ of banks:

Proposition 2. *Consider a bank's portfolio allocation problem as in 19. Then:*

1. *The exists a unique solution $\boldsymbol{x}_b^*(\boldsymbol{p}, \boldsymbol{r}_b, \boldsymbol{\sigma}^2, X_{!b})$.*
2. *x_{bf}^* is non-decreasing in r_{bf}, and non-increasing in $r_{bf'}$. Consequently, it is non-increasing in τ_{bf} and non-decreasing in $\tau_{bf'}$.*

3. *The share $x_{b'b}$ is non-increasing with $\tau_{b'b}$ for any bank b'.*

Proof. See Appendix A.2 □

The proposition is fairly standard and establishes the behaviour of banks with respect to tax instruments. If the average return of an asset increases, banks will invest more in it. Since tax rates influence this average return, it can be used to change the investment decision of banks.

The **market equilibrium** of this model is a set of quantities $x^*_{ba}(\boldsymbol{p}^*, X^*_{!b})$ and a vector of prices $\boldsymbol{p}^*$ such that banks maximize their payoff (as in 19) and the allocation is feasible for all assets: $\sum_b x^*_{ba} \leq 1 \; \forall a$. Because the model is concave in $X_{!b}, \boldsymbol{p}$ for all banks, the single crossing property in these arguments applies and the market equilibrium is unique.

It should be noted that a market equilibrium is not necessarily market clearing. As the discussion in the previous paragraphs highlighted, the condition on X_B that is required to have any discussion about an equilibrium is that all of its eigenvalues are smaller than one. This implies in turn that at least one column of X_B is smaller than one, meaning that not all shares of a given bank are sold. This is so because for any arbitrarily low price, buying shares of a bank increases the amplification on variance faster than the amplification on the returns and therefore the payoff maximization becomes unconstrained by the budget. Hence, prices do not clear the interbanking network and some shareholders who are not banks would be required for market clearing.

2.3 Endogenous Asymptotics: The Curse of Asymmetry

It should be fairly intuitive that in an idealized economy, productivity attracts investments. In the context of this model, banks with an higher ability to extract returns from fundamental assets can themselves provide higher returns to other investors. So in equilibrium, these banks becomes attractors for other banks.

This simple idea has an important consequence in terms of risk amplification: because productive banks concentrate investment, they also concentrate risk. This asymmetry between attractive banks and other banks thus exacerbate the network externality. When banks are identical and able to extract an identical return for every asset, the network becomes symmetric. In this context, an egoistic payoff maximization accounts for all other decisions because they are identical. In other cases, the externality remains and the higher is the asymmetry, the higher is the externality (regardless of the size of the banking sector).

To convey the intuition, I restrict the model in the main text to only two types of banks, that is productive banks and non-productive banks. These two types of bank can be thought as the supremum and the infemum of larger population of banks, as in the general proof in appendix. The general result, extending this discussion to the full model, is proven in appendix A.3. Productive banks have an higher ability r_H to extract profits out of fundamental assets, meaning that they can offer higher returns on other investors. In comparison, non-productive banks can only extract r_L from fundamental assets.

As there are only two types of banks, the endogenous topology reduces to four types of connections on the interbanking network: from attractive banks to attractive banks (valued at x^*_{HH}), from regular banks to attractive banks (valued at x^*_{LH}), from regular banks to regular banks (valued at x^*_{LL}) and from attractive banks to regular banks (valued at x^*_{HL}). These numbers are derived from the banks' problem and their multiplicity arise from the symmetry of the simplified problem.

Denote α and $(1-\alpha)$ respectively the fraction of attractive banks and unattractive banks in the economy. It is fairly straightforward to show that the endogenous shocks for each type of banks is then given by:

$$\epsilon_H = \frac{(1-(1-\alpha)x^*_{LL})\alpha \sum_f x^*_{Hf}\epsilon_f + \alpha(1-\alpha)x^*_{HL}\sum_f x^*_{Lf}\epsilon_f}{(1-\alpha x^*_{HH})(1-(1-\alpha)x^*_{LL}) - \alpha(1-\alpha)x^*_{HL}x^*_{LH}}, \tag{21}$$

$$\epsilon_L = \frac{(1-\alpha x^*_{HH})(1-\alpha)\sum_f x^*_{Lf}\epsilon_f + \alpha(1-\alpha)x^*_{LH}\sum_f x^*_{Hf}\epsilon_f}{(1-(1-\alpha)x^*_{LL})(1-\alpha x^*_{HH}) - \alpha(1-\alpha)x^*_{HL}x^*_{LH}}, \tag{22}$$

where the inequalities $1 > \alpha x_{HH}, 1 > (1-\alpha)x_{LL}, 1 > (1-\alpha x^*_{LL})(1-(1-\alpha)x^*_{HH}) - (1-\alpha)x^*_{LH}\alpha x_{HL}*$ hold.

If we focus on the case where α approaches one, meaning that attractive banks become predominant in the economy, equation 21 approaches:

$$\epsilon_H = \frac{\sum_f x^*_{Hf}\epsilon_f}{1-x^*_{HH}}, \tag{23}$$

which depends solely on bank H's decisions. So as the predominance of attractive banks increases, the decisions made by other banks bear little impact and eventually vanishes. It is fearly easy to check that in such context, the endogenous decision of bank H will be such that $\frac{\sum_f x^*_{HF}\epsilon_f}{(1-x^*_{HH})} = \sum_f \epsilon_f$. This is the case because the average return of bank H and the sum of average returns on fundamental assets are identical. Hence, the payoff maximization commands a minimization of the variance, which occurs when there is no amplification on the inter banking network.

The fact that the externality vanishes does not depend on the type of bank *per se*, as similar results can be found for non attractive banks (e.g.: when $(1-\alpha)$ goes to 1). It is rather on the homogeneity of banks that leads to a regular (weighted) graph. In an economy where attractive banks can buy non-attractive ones, so as to replicate its own ability to extract returns, the inter banking network would then converge to a first best. In a perhaps less intuitive story, if attractive banks where to fail more often than non-attractive banks, the economy would also converge to an optimum without externalities.

If however both type of banks coexist, the inter banking network generates an sub-optimal allocation of ressources which can be improved upon by tax instruments. In appendix[3], I generalise the result to the general set-up by proving the following proposition:

Proposition 3. *Let $F(\boldsymbol{\sigma^2})$ be a finite set of fundamental assets and*

$$\mathcal{B} \equiv \{b_1(\boldsymbol{r}_1, f(b_1)), \ldots, b_n(\boldsymbol{r}_n, f(b_n))\}$$

be the set of bank profiles defining returns and ownership on those fundamental assets. Define further $\mathbb{1}(b) \in \mathcal{B}^{|B|}$ a profile of identical banks b. Consider an equilibrium topology $X^*(B,F))$ as defined in the previous pages. Then, $X^*(B,F)$ shows no externality if and only if there exists a b such that $\mathbb{1}(b) = B$. In such case, $\sigma_b^2 = \sum_f \sigma_f^2 \ \forall b \in B$.

Proof. See Appendix A.3 □

The result relates to theorem 4 in (Acemoglu, Carvahlo, Ozdaglar & Tabahz-Salehi, 2012) about "balanced graphs" (theorem 4). They show that the redistributive externality[4] vanishes to zero as the size of the network increases. In their world, as the number of balanced connections increases, the variance decreases at the rate $n^{-1/2}$. This is so because shocks are independent and generated by the number of growing of production nodes (which would here be the fundamental assets). In this paper, however, there is conservation of variance for two reasons. First, the number of fundamental assets remains fixed. Second, banks control the share of assets they put in their portfolio, whereas in their paper the matrix X is fixed. As some positive shares are required to generate any returns at all, the input in variance is made proportionate to returns.

[3] I still need to write it, but it should be straightforward

[4] In their paper, the impact of the network on business shocks is given by $\alpha(I-(1-\alpha)X)^{-1}$, where X is a stochastic matrix. As such, the leading eigenvalue of X is one and thus, the (unique) leading eigenvalue of $\alpha(I-(1-\alpha)X)^{-1}$ also equals one. Hence, the amplification effect is null and only the redistributive aspect is analyzed.

2.3.1 Asymmetry in Returns

The previous section discussed the magnitude of the externality in terms of the shares of banks with different profiles on assets. As the share of one profile took over the economy, I argued that the externality vanished. In this section, I let the shares fixed and rather increase the returns of one single bank. By doing so, the magnitude of the externality increases as well.

To stick with the example, if the most attractive bank can provide even higher returns, the less attractive bank will rely more and more on the interbanking connections to extract profits, as stated in proposition 2. But by doing so, they increase amplification.

The proof for this is quite simple and can be understood through the analysis of the standard eigenvalue/egeinvector equation. Consider the solution for the leading eigenvalue:

$$\lambda v_{b'} = \sum_{b} x^*_{b'b} v_b \ \forall b' \in B. \tag{24}$$

Now, assume that b' is the bank with the highest returns and assume an increase in such return. The previous equation tells us that:

$$\left(\sum_{b,b'} \frac{\partial \lambda}{\partial x^*_{b'b}} \frac{\partial x^*_{b'b}}{\partial r_H} \right) v_{b'} + \lambda \sum_{b} \frac{\partial v_{b'}}{\partial x^*_{b'b}} \frac{\partial x^*_{b'b}}{\partial r_H} = \sum_{b} \frac{\partial x_{b'b}}{\partial r_H} v_b + \sum_{b} x_{bb'} \sum_{b'} \frac{\partial v_b}{\partial x_{bb'}} \frac{\partial x_{bb'}}{\partial r_H} \tag{25}$$

Recall that eigenvectors are defined up to a scale, meaning that they have one free parameter (usually fixed by letting $||v|| = 1$). This imply that the sum of changes on eigenvectors sum to zero. Hence, summing on both sides yield:

$$|B| \left(\sum_{b,b'} \frac{\partial \lambda}{\partial x^*_{b'b}} \frac{\partial x^*_{b'b}}{\partial r_H} \right) v_{b'} = \sum_{b,b'} \frac{\partial x_{b'b}}{\partial r_H} v_b. \tag{26}$$

From proposition 2, we know that derivatives with respect to r_H are positive. Further more, the components of the eigenvector associated with the largest eigenvalue are all positive. Thus, the change in the largest eigenvalue increases.

3 Optimal Taxation in a Static Setup

I now turn on the task of solving the risk externality. I begin this by assuming that a benevolent government seeks to solve the optimal inter banking network given that it controls all the investment decisions.

In the section that comes afterwards, I move on to a decentralized framework where the government can only use tax instruments to influence the decentralized market equilibrium.

The key result is as follows: to reach a first best, the government requires at least as many tax instruments as there are different values of r_{ba}. If not, the government can improve welfare, but only reach a second best.

3.1 A Centralized Economy

In this framework, a central planner maximizes the sum of payoffs of banks and solve:

$$\boldsymbol{X}^{opt} \equiv \arg \max_{\{x_{ba}|\ \forall b \in B, a \in A\}} \sum_b \omega_b u(\boldsymbol{x}_b, \tilde{\boldsymbol{r}}_{\boldsymbol{b}}, \boldsymbol{\sigma}^2, X_{!b})$$
$$\text{s.t.} \sum_b x_{ba} \leq 1\ \forall a \in A \tag{27}$$

This means that the government cares for all portfolios, and each portfolio is weighted according to ω_b (perhaps measuring the number of investors). This problem is globally concave and has therefore a unique solution. Denote L_a the Lagrange multiplier associated with the constraint on asset a. The optimal quantities are given by the first order conditions, which can be arranged in the following form:

$$L_a - \sum_{b' \in B} \omega_{b'} \left[\sum_{c \in B} \frac{\partial \tilde{r}_{b'c}}{\partial x_{ba}} x_{b'c} - h'_{b'}(\cdot) \frac{\partial \sigma_c^2}{\partial x_{ba}} \frac{x_{b'c}^2}{2} \right] = \omega_b \left[\tilde{r}_{ba} - h'_b(\cdot)_b \sigma_a^2 x_{ba} \right]\ \forall a \in A, b \in B, \tag{28}$$

The contrasting difference between this equation and the first order condition of a single bank is that the government accounts for the whole network effect of investing in a, as measured by the second term in the left hand side. In a single bank, only on term in the summation is accounted for, that is the term that accounts for the effects on the bank itself. When the government decides, it also account for the accrued variance in other banks. Since the left hand side is identical for any bank b, it implies that each bank gets the same (weighted) marginal payoff on asset a. In other words, the government behaves as a portfolio manager facing no cross correlations or externalities, but has a larger shadow price for asset on the interbanking network.

3.2 A Decentralized Economy

3.2.1 Effect of Taxes on Payoff

Let V_b^* be the indirect payoff function of bank b. A variation of the tax rate $\tau_{dc}, d \in B, c \in A$, can be decomposed in the following fashion:

$$\frac{\partial V_b^*}{\partial \tau_{dc}} = \underbrace{-\mathbb{1}_{d=b}\tilde{r}_{bc}^* x_{bc}^*}_{\text{Direct Effect}} - \underbrace{L_b \sum_a p_a \frac{dx_{ba}^*}{d\tau_{dc}}}_{\text{Value Effect}} \dots$$

$$\dots + \sum_a x_{ba}^* \underbrace{\left[\frac{dr_{ba}^*}{d\tau_{dc}} - \sum_{x_{ba'} \in \boldsymbol{x}_b} \frac{\partial r_{ba}}{\partial x_{ba'}^*} \frac{dx_{ba'}^*}{d\tau_{dc}} \right]}_{\text{Return Externality}} \dots$$

$$\dots - h_b' \sum_a \frac{x_{ba}^{*2}}{2} \underbrace{\left[\frac{d\sigma_a^{2*}}{d\tau_{dc}} - \sum_{x_{ba'} \in \boldsymbol{x}_b} \frac{\partial \sigma_a^{*2}}{\partial x_{ba'}^*} \frac{dx_{ba'}^*}{d\tau_{dc}} \right]}_{\text{Risk Externality}}, \tag{29}$$

where $\mathbb{1}_{d=b}$ is the indicator function, and $\frac{d(\cdot)}{d\tau_d c}$ is the total derivative[5] of a variable with respect to τ_{dc}.

The variation in payoff can be understood through the decomposition of four terms. The first term measures the direct effect that the tax instrument has on returns and is present only if the tax is on the As taxation decreases returns, it is necessarily negative. The second term measures the impact of taxation on the budget constraint. If it is binding, it measures the change in value of the portfolio, measured in units of payoff.

The two other terms measure the network externality. The first term accounts for the the impact of the tax on returns of each asset of the portfolio. The second account for the impact of the tax on the variance of each asset. Both are expressed as a difference between the total effect if the tax and the effect that is accounted by banks in choosing their allocation. In other words, the impact of the externality of the payoff is the impact through the investment decisions of other banks.

[5]See the Appendix A.4 for formal definitions.

3.3 Government Problem

4 Dynamic Cascades and Taxation

5 Empirical Example

To do.

A Proof of Various Sections

A.1 Proof of Proposition 1

Proof. 1. It is sufficient to notice that $\boldsymbol{\epsilon}(X_B, X_F)$ is an infinite power of X_B. Each component is therefore the limit of a sum of products of powers of some of the components of X_B:

$$\epsilon_b(X_B, X_F) = \lim_{n \to \infty} \sum_{a \in A} x_{ba} [X_B^{n-1}]_{ab}$$

As all these sum of powers are positive, it follows that the derivative with respect to any variable x_{ba} is also positive.

2. Notice that:

$$\boldsymbol{\epsilon}_B = (I - X_B)^{-1} X_F \boldsymbol{\epsilon}_F,$$

which explicits that ϵ_B is linear in any element in X_F. If there is a path between b and f, then $[(I - X_B)^{-1} X_F]_{bf}$ is non-zero as the element is a discounted measure of all the paths from f to b . For any of these paths, it is linear in its first elements $x_{b'f}$ since it belongs in X_F.

□

A.2 Proof of Proposition 2

Proof.
For the most part, this is a textbook proof. It is useful to begin with a de-

scription of the bordered hessian, which has the following signs for cofactors:

$$\text{sign}\left(\text{Cof}(H)\right) \equiv \begin{bmatrix} - & \text{sign}(-1)^{|A|-1} & \text{sign}(-1)^{|A|-1} & \dots & \text{sign}(-1)^{|A|-1} \\ \text{sign}(-1)^{|A|-1} & \text{sign}(-1)^{|A|} & \text{sign}(-1)^{|A|-1} & \dots & \text{sign}(-1)^{|A|-1} \\ \text{sign}(-1)^{|A|-1} & \text{sign}(-1)^{|A|-1} & \text{sign}(-1)^{|A|} & \dots & \text{sign}(-1)^{|A|-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \text{sign}(-1)^{|A|-1} & \text{sign}(-1)^{|A|-1} & \text{sign}(-1)^{|A|-1} & \dots & \text{sign}(-1)^{|A|} \end{bmatrix}. \tag{30}$$

This allows to sign derivatives with ease in the remainder of the proof.

1. The payoff functions are quadratic in $(X_F'(1-X_B)^{-1}\boldsymbol{x}_b, \boldsymbol{x}_f)$. There is thus a unique solution vector $(X_F'(1-X_B)^{-1}\boldsymbol{x}_b, \boldsymbol{x}_f)$ to the bank's problem. Because there is a one to one correspondance with $\boldsymbol{x}_B, \boldsymbol{x}_F$, the solution is unique.

2. Because the problem is quadratic, the bordered hessian matrix H of the problem is negative semi-definite. Hence, the determinant of each principal minor alternates in sign while all minors associated with the bank's variable are negative if $|A|$ is even and positive if odd. For the derivative of ownership $x_{bk}, k \in A$ with respect to the return of asset f (r_{bf}), the implicit function is given by the system of equations by derivation of the first order conditions:

$$-D = H\nabla \boldsymbol{x}_b^*, \tag{31}$$

where $\nabla \boldsymbol{x}_b^*$ is a vector of derivatives of endogenous variables (including the lagrange multiplier), H is the bordered hessian and D are the direct effects of the variable on the first order conditions. Using Cramer's rule, the sign of $\frac{\partial x_{bf}}{\partial r_{bf}}$ is given by:

$$\text{sign}\left(\frac{\partial x_{bf}}{\partial r_{bf}}\right) = \text{sign}\left(\frac{\text{det}H_f}{\text{det}H}\right), \tag{32}$$

where H_f is the column f is substituted by the negative of the direct effect vector (LHS of 31).

If $|A|$ is even, the determinant of H is negative while it is positive if odd. Because the direct effect vector is equal to -1 only in the f-th position and zero elsewhere, the determinant of H_f is equivalent to the cofactor at position f, f multiplied by -1. As they are all of opposite side of the determinant, it makes the ratio positive.

Following the same logic, the sign of the derivative of ownership in asset $f' \neq f$ is given by:

$$\text{sign}\left(\frac{\partial x_{bf'}}{\partial r_{bf}}\right) = \text{sign}\left(\frac{\det H_{f'}}{\det H}\right). \tag{33}$$

For each other position different than f, the sign of the minor associated with $H_{f'}$ is of the same sign of $\det H$ and thus $\det H_f$ is positive, which implies a negative sign on the derivative.

As for the derivatives with respect to the tax rates, it is sufficient to notice that $\frac{\partial x_{bf}}{\partial \tau_{bf'}} = -\frac{\partial x_{bf}}{\partial r_{bf'}}$ for all $f, f' \in F$.

3. Notice that the direct effect vector has the form:

$$D = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ -\tilde{r}_{ba} - \frac{\partial \tilde{r}_{ba}}{\partial x_{ba}} \\ \\ \vdots \\ 0 \end{bmatrix} \tag{34}$$

The same logic applies as in the previous point by using this vector in 31.

□

A.3 Proof of Proposition 3

A.4 Variation of Indirect Utility With Tax Rates

The equilibrium payoff of bank b is given by:

$$V_b^* \equiv \sum_a r_{ba}^* x_{ba}^* - h\left(\sum_a \sigma_a^{*2} \frac{x_{ba}^{*2}}{2}\right) \tag{35}$$

The variation of payoffs with respect to tax the rate on the return of bank d on asset c is given by:

$$\frac{\partial V_b^*}{\partial \tau_{dc}} = - \mathbb{1}_{d=b}\tilde{r}_{bc}^* x_{bc}^* + \sum_a \left[r_{ba}^* - h_b'(\cdot)\sigma_a^{*2} x_{ba}^* \right] \left[\frac{\partial x_{ba}^*}{\partial \tau_{dc}} + \sum_{a''} \frac{\partial x_{ba}^*}{\partial p_{a''}} \frac{\partial p_{a''}}{\partial \tau_{dc}} \right] \dots$$
$$\dots + \sum_a \sum_{b'a'} \left[x_{ba}^* \frac{\partial r_{ba}}{\partial x_{b'a'}^*} - h_b'(\cdot) \frac{x_{ba}^{*2}}{2} \frac{\partial \sigma_a^{*2}}{\partial x_{b'a'}^*} \right] \left[\frac{\partial x_{b'a'}^*}{\partial \tau_{dc}} + \sum_{a''} \frac{\partial x_{b'a'}}{\partial p_{a''}} \frac{\partial p_{a''}}{\partial \tau_{dc}} \right], \tag{36}$$

where $\mathbb{1}_{d=c}$ is the indicator function on wheter $d = b$ or not.

Denote the total derivative by:

$$\frac{dx_{b'a'}^*}{d\tau_{dc}} \equiv \frac{\partial x_{b'a'}^*}{\partial \tau_{dc}} + \sum_{a''} \frac{\partial x_{b'a'}}{\partial p_{a''}} \frac{\partial p_{a''}}{\partial \tau_{dc}}, \tag{37}$$

$$\frac{dr_{ba}^*}{d\tau_{dc}} \equiv \sum_{x_{b'a'}^* \in X^*} \frac{\partial r_{ba}}{\partial x_{b'a'}^*} \frac{dx_{b'a'}^*}{d\tau_{dc}}, \tag{38}$$

$$\frac{d\sigma_a^{2*}}{d\tau_{dc}} \equiv \sum_{x_{b'a'}^* \in X^*} \frac{\partial \sigma_a^{2*}}{\partial x_{b'a'}^*} \frac{dx_{b'a'}^*}{d\tau_{dc}}, \tag{39}$$

that is the total effect of τ_{dc} has on a given variable through the direct effect on equilibrium values and through the effect on equilibrium prices.

Using the first order condition on bank b and these definitions, this equation can be reduced to:

$$\frac{\partial V_b^*}{\partial \tau_{dc}} = \underbrace{- \mathbb{1}_{d=b}\tilde{r}_{bc}^* x_{bc}^*}_{\text{Direct effect}} \underbrace{- L_b \sum_a p_a \frac{dx_{ba}^*}{d\tau_{dc}}}_{\text{Price Effect}} \dots$$
$$\dots + \sum_a x_{ba}^* \underbrace{\left[\frac{dr_{ba}^*}{d\tau_{dc}} - \sum_{x_{b'a'} \in \boldsymbol{x}_b} \frac{\partial r_{ba}}{\partial x_{b'a'}^*} \frac{dx_{b'a'}^*}{d\tau_{dc}} \right]}_{\text{Return Externality}} \dots$$
$$\dots - h_b' \sum_a \frac{x_{ba}^{*2}}{2} \underbrace{\left[\frac{d\sigma_a^{2*}}{d\tau_{dc}} - \sum_{x_{b'a'} \in \boldsymbol{x}_b} \frac{\partial \sigma_a^{*2}}{\partial x_{b'a'}^*} \frac{dx_{b'a'}^*}{d\tau_{dc}} \right]}_{\text{Risk Externality}}. \tag{40}$$

B Old Stuff

B.1 Proof that V_0 is Convex (sketch)

1. The probability of transition to state s' from state s without a cascade, is given by:

$$PS(s'|s) = \int_{-a_{f_1}}^{a_{f_1}} \dots \int_{-a_k} \int_{\max\{c_{s1}(x),c_{s2}(x),\dots,c_{sk}(x)\}}^{\min\{c_{f1}(x),c_{f2}(x),\dots,c_{fk}(x)\}} d\epsilon \dots d\epsilon$$

   This is a concave function in x. The probability for transition from cascades is similar.

2. The product of two concave functions is concave if the Gonzi condition is satisfied:

$$-(u(x^1) - u(x^2)(P(x^1) - P(x^2)) \ \forall x^1, x^2$$

3. Consider the value fonction V_0:

$$V_0(x) = u(x) + \beta \sum_{s \in S} P_s(x) V_s(x)$$

   (skipping the delta notation and loosely use diffs'). Difff with respect to x_{ij} yields:

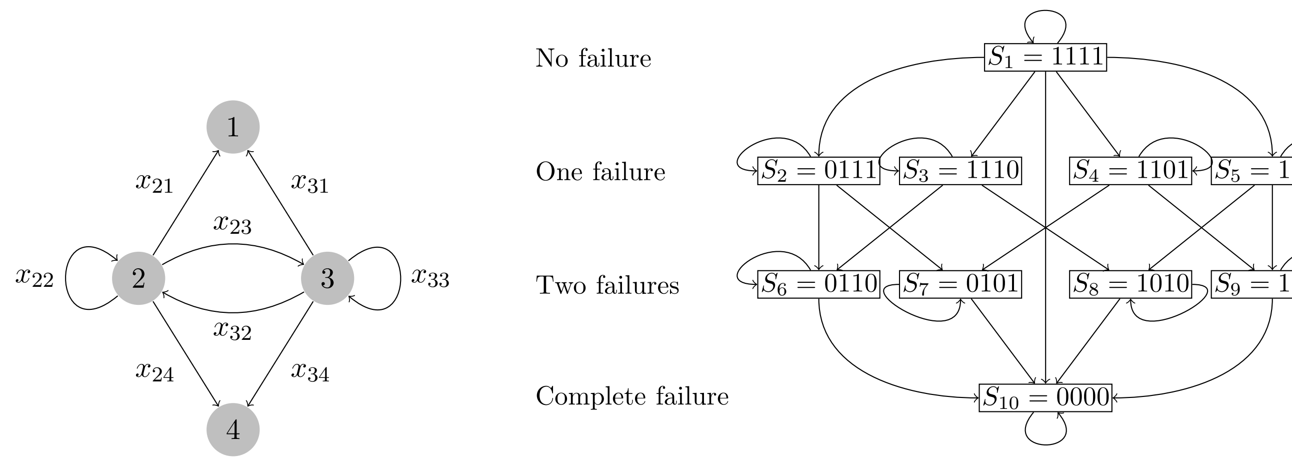
$$V'(x) = u'(x) \left[1 + \sum_{s \in S(s_x=1)} P_s(x) \right] - \beta \sum_{s \in S} (V_0 - V_s) \frac{\partial P_s}{\partial x}$$

   Consider the smallest element $V_0 - V_s$, call it v_{small}, one then has:

$$\begin{aligned} -\beta \sum_{s \in S} (V_0 - V_s)(P_s(x^1) - P_s(x^2) &\leq v_{small} \sum_{s} \left[P_s(x^1) - P_s(x^2) \right] \\ &= -v_{small}(\Delta P_{00}(x^1) - \Delta P_{00}(x^2)) \end{aligned}$$

Now, denote the stochastic matrix $P_c \in 2^{2N}$ of cascades and index states by i, j. A typical component of P_c is given by:

$$P_{ij} = \begin{cases} P_{1k}(\boldsymbol{s}_i | \boldsymbol{s}_j) & \text{if } i \in S_{k+1}, j \in S_k \text{ for some } k, \\ 0 & \text{otherwise.} \end{cases} \tag{41}$$



(a) **Two Assets (1 & 4) and two Banks (2 & 3)**

(b) **Various States of the Networks (Some Transitions Not S**

Figure 1: **The Example Used Throughout This Paper**

By construction, P is a stochastic matrix and the probability of $P(\boldsymbol{s}_i|\boldsymbol{s}_j)$ of a cascade for any two given states is given by the ij-th component of:

$$P^* = \lim_{n\to\infty} P^n \tag{42}$$

Following the lines of proposition XX:

Proposition 4. *The components $P^*_{ij}(X)$ are convex in $P_{ij}(R(X, s_j))$.*

C A Simple Example

C.1 Full Knowledge of Strategies

In this section, I solve a very simple example with two banks and two assets (see Figure 1(a)). I present in Figure 1(b) the associated network of states. Each arrow in this network gives a possible transition from one state to another. For the clarity of the Figure, all transitions where there are two simultaneous failures are not shown.

The partial order of relation is also explicited through levels of The value function of bank number one in state i with strategy $\boldsymbol{x}_2 \equiv [x_{21}, x_{22}, x_{23}, x_{24}]'$, assuming a strategy $\boldsymbol{x}_3$ from the other bank, is given by:

$$V_1(S_i, \boldsymbol{x}_2, \boldsymbol{x}_3) \equiv \mathbb{E}\left[\boldsymbol{x}_1 | S_i\right] + \beta \sum_{S_j} P(S_j | S_i, \boldsymbol{x}_2, \boldsymbol{x}_3) V_1(S_j, \boldsymbol{x}_2, \boldsymbol{x}_3),$$

where the term $\mathbb{E}\left[\boldsymbol{x}_2 | S_1\right]$ denotes the expected utility of $\boldsymbol{x}_2$ given the current state (1). The term $P(S_j | S_i, \boldsymbol{x}_2, \boldsymbol{x}_3)$ is the probability of transition from state i to state j. For simplicity, these probabilities will be abbreviated as $P_{ji}(\boldsymbol{x})$, forming the matrix $P(\boldsymbol{x})$:

$$P(\boldsymbol{x}) \equiv \begin{bmatrix} P_{11}(\boldsymbol{x}) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ P_{21}(\boldsymbol{x}) & P_{22}(\boldsymbol{x}) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ P_{31}(\boldsymbol{x}) & 0 & P_{33}(\boldsymbol{x}) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ P_{41}(\boldsymbol{x}) & 0 & 0 & P_{44}(\boldsymbol{x}) & 0 & 0 & 0 & 0 & 0 & 0 \\ P_{51}(\boldsymbol{x}) & 0 & 0 & 0 & P_{55}(\boldsymbol{x}) & 0 & 0 & 0 & 0 & 0 \\ P_{61}(\boldsymbol{x}) & P_{62}(\boldsymbol{x}) & P_{63}(\boldsymbol{x}) & 0 & 0 & P_{66}(\boldsymbol{x}) & 0 & 0 & 0 & 0 \\ P_{71}(\boldsymbol{x}) & P_{72}(\boldsymbol{x}) & 0 & P_{74}(\boldsymbol{x}) & 0 & 0 & P_{77}(\boldsymbol{x}) & 0 & 0 & 0 \\ P_{81}(\boldsymbol{x}) & 0 & P_{83}(\boldsymbol{x}) & 0 & P_{85}(\boldsymbol{x}) & 0 & 0 & P_{88}(\boldsymbol{x}) & 0 & 0 \\ P_{91}(\boldsymbol{x}) & 0 & 0 & P_{94}(\boldsymbol{x}) & P_{95}(\boldsymbol{x}) & 0 & 0 & 0 & P_{99}(\boldsymbol{x}) & 0 \\ P_{101}(\boldsymbol{x}) & P_{102}(\boldsymbol{x}) & P_{103}(\boldsymbol{x}) & P_{104}(\boldsymbol{x}) & P_{105}(\boldsymbol{x}) & P_{106}(\boldsymbol{x}) & P_{107}(\boldsymbol{x}) & P_{108}(\boldsymbol{x}) & P_{109}(\boldsymbol{x}) & 1 \end{bmatrix}$$

This means that the system has the following solution:

$$\boldsymbol{V}_1(\boldsymbol{x}) = \left(I - \beta P(\boldsymbol{x})\right)^{-1} \boldsymbol{E}(\boldsymbol{x}),$$

with the obvious vectorial equivalent for $\boldsymbol{V}$ and $\boldsymbol{E}$

Since the system has a neat lattice form, the terminal conditions of the states with two failures are given by:

$$\begin{aligned} V_1(S_6, \boldsymbol{x}_2, \boldsymbol{x}_3) &= \frac{\mathbb{E}\left[\boldsymbol{x}_2 | S_6\right]}{1 - \beta P_{66}(\boldsymbol{x})}, \\ V_1(S_7, \boldsymbol{x}_2, \boldsymbol{x}_3) &= \frac{\mathbb{E}\left[\boldsymbol{x}_2 | S_6\right]}{1 - \beta P_{77}(\boldsymbol{x})}, \\ V_1(S_8, \boldsymbol{x}_2, \boldsymbol{x}_3) &= \frac{\mathbb{E}\left[\boldsymbol{x}_2 | S_6\right]}{1 - \beta P_{88}(\boldsymbol{x})}, \\ V_1(S_9, \boldsymbol{x}_2, \boldsymbol{x}_3) &= \frac{\mathbb{E}\left[\boldsymbol{x}_2 | S_6\right]}{1 - \beta P_{99}(\boldsymbol{x})}. \end{aligned}$$

with, of course, $V_1(S_{10}) = 0$.

Recursive substitution leads to:

$$V_1(S_2, \boldsymbol{x}_2, \boldsymbol{x}_3) = \left[\frac{\mathbb{E}(\boldsymbol{x}_2|S_2)}{(1-\beta P_{22}(\boldsymbol{x}))} + \frac{\beta P_{62}(\boldsymbol{x})}{(1-\beta P_{22}(\boldsymbol{x}))} \frac{\mathbb{E}\left[\boldsymbol{x}_2|S_6\right]}{(1-\beta P_{66}(\boldsymbol{x}))} + \frac{\beta P_{72}(\boldsymbol{x})}{(1-\beta P_{22}(\boldsymbol{x}))} \frac{\mathbb{E}\left[\boldsymbol{x}_2|S_7\right]}{(1-\beta P_{77}(\boldsymbol{x}))} \right],$$

$$V_1(S_3, \boldsymbol{x}_2, \boldsymbol{x}_3) = \left[\frac{\mathbb{E}(\boldsymbol{x}_2|S_3)}{(1-\beta P_{33}(\boldsymbol{x}))} + \frac{\beta P_{63}(\boldsymbol{x})}{(1-\beta P_{33}(\boldsymbol{x}))} \frac{\mathbb{E}\left[\boldsymbol{x}_2|S_6\right]}{(1-\beta P_{66}(\boldsymbol{x}))} + \frac{\beta P_{83}(\boldsymbol{x})}{(1-\beta P_{33}(\boldsymbol{x}))} \frac{\mathbb{E}\left[\boldsymbol{x}_2|S_8\right]}{(1-\beta P_{88}(\boldsymbol{x}))} \right],$$

$$V_1(S_4, \boldsymbol{x}_2, \boldsymbol{x}_3) = \left[\frac{\mathbb{E}(\boldsymbol{x}_2|S_4)}{(1-\beta P_{44}(\boldsymbol{x}))} + \frac{\beta P_{74}(\boldsymbol{x})}{(1-\beta P_{44}(\boldsymbol{x}))} \frac{\mathbb{E}\left[\boldsymbol{x}_2|S_7\right]}{(1-\beta P_{77}(\boldsymbol{x}))} + \frac{\beta P_{94}(\boldsymbol{x})}{(1-\beta P_{44}(\boldsymbol{x}))} \frac{\mathbb{E}\left[\boldsymbol{x}_1|S_9\right]}{(1-\beta P_{99}(\boldsymbol{x}))} \right],$$

$$V_1(S_5, \boldsymbol{x}_2, \boldsymbol{x}_3) = \left[\frac{\mathbb{E}(\boldsymbol{x}_2|S_5)}{(1-\beta P_{55}(\boldsymbol{x}))} + \frac{\beta P_{65}(\boldsymbol{x})}{(1-\beta P_{55}(\boldsymbol{x}))} \frac{\mathbb{E}\left[\boldsymbol{x}_2|S_6\right]}{(1-\beta P_{66}(\boldsymbol{x}))} + \frac{\beta P_{95}(\boldsymbol{x})}{(1-\beta P_{55}(\boldsymbol{x}))} \frac{\mathbb{E}\left[\boldsymbol{x}_1|S_9\right]}{(1-\beta P_{99}(\boldsymbol{x}))} \right].$$

Which means:

$$\begin{aligned}
V_1(S_1, \boldsymbol{x}_2, \boldsymbol{x}_3) = {} & \frac{\mathbb{E}(\boldsymbol{x}_2|S_1)}{(1-\beta P_{11}(\boldsymbol{x}))} \cdots \\
& + \frac{\beta P_{21}(\boldsymbol{x})}{(1-\beta P_{11}(\boldsymbol{x}))} \left[\frac{\mathbb{E}(\boldsymbol{x}_2|S_2)}{(1-\beta P_{22}(\boldsymbol{x}))} + \frac{\beta P_{62}(\boldsymbol{x})}{(1-\beta P_{22}(\boldsymbol{x}))} \frac{\mathbb{E}\left[\boldsymbol{x}_2|S_6\right]}{(1-\beta P_{66}(\boldsymbol{x}))} + \frac{\beta P_{72}(\boldsymbol{x})}{(1-\beta P_{22}(\boldsymbol{x}))} \frac{\mathbb{E}\left[\boldsymbol{x}\right.}{(1-\beta} \right. \\
& + \frac{\beta P_{31}(\boldsymbol{x})}{(1-\beta P_{11}(\boldsymbol{x}))} \left[\frac{\mathbb{E}(\boldsymbol{x}_2|S_3)}{(1-\beta P_{33}(\boldsymbol{x}))} + \frac{\beta P_{63}(\boldsymbol{x})}{(1-\beta P_{33}(\boldsymbol{x}))} \frac{\mathbb{E}\left[\boldsymbol{x}_2|S_6\right]}{(1-\beta P_{66}(\boldsymbol{x}))} + \frac{\beta P_{83}(\boldsymbol{x})}{(1-\beta P_{33}(\boldsymbol{x}))} \frac{\mathbb{E}\left[\boldsymbol{x}\right.}{(1-\beta} \right. \\
& + \frac{\beta P_{41}(\boldsymbol{x})}{(1-\beta P_{11}(\boldsymbol{x}))} \left[\frac{\mathbb{E}(\boldsymbol{x}_2|S_4)}{(1-\beta P_{44}(\boldsymbol{x}))} + \frac{\beta P_{74}(\boldsymbol{x})}{(1-\beta P_{44}(\boldsymbol{x}))} \frac{\mathbb{E}\left[\boldsymbol{x}_2|S_7\right]}{(1-\beta P_{77}(\boldsymbol{x}))} + \frac{\beta P_{94}(\boldsymbol{x})}{(1-\beta P_{44}(\boldsymbol{x}))} \frac{\mathbb{E}\left[\boldsymbol{x}\right.}{(1-\beta} \right. \\
& + \frac{\beta P_{51}(\boldsymbol{x})}{(1-\beta P_{11}(\boldsymbol{x}))} \left[\frac{\mathbb{E}(\boldsymbol{x}_2|S_5)}{(1-\beta P_{55}(\boldsymbol{x}))} + \frac{\beta P_{65}(\boldsymbol{x})}{(1-\beta P_{55}(\boldsymbol{x}))} \frac{\mathbb{E}\left[\boldsymbol{x}_2|S_6\right]}{(1-\beta P_{66}(\boldsymbol{x}))} + \frac{\beta P_{95}(\boldsymbol{x})}{(1-\beta P_{55}(\boldsymbol{x}))} \frac{\mathbb{E}\left[\boldsymbol{x}\right.}{(1-\beta} \right. \\
& + \frac{\beta P_{61}(\boldsymbol{x})}{(1-\beta P_{11}(\boldsymbol{x}))} \frac{\mathbb{E}\left[\boldsymbol{x}_2|S_6\right]}{1-\beta P_{66}(\boldsymbol{x})} + \frac{\beta P_{71}(\boldsymbol{x})}{(1-\beta P_{11}(\boldsymbol{x}))} \frac{\mathbb{E}\left[\boldsymbol{x}_2|S_7\right]}{1-\beta P_{77}(\boldsymbol{x})} \cdots \\
& + \frac{\beta P_{81}(\boldsymbol{x})}{(1-\beta P_{11}(\boldsymbol{x}))} \frac{\mathbb{E}\left[\boldsymbol{x}_2|S_8\right]}{1-\beta P_{88}(\boldsymbol{x})} + \frac{\beta P_{91}(\boldsymbol{x})}{(1-\beta P_{11}(\boldsymbol{x}))} \frac{\mathbb{E}\left[\boldsymbol{x}_2|S_9\right]}{1-\beta P_{99}(\boldsymbol{x})}.
\end{aligned}$$