FINANCIAL LINKAGES, PORTFOLIO CHOICE AND SYSTEMIC RISK

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Financial linkages smooth the shocks faced by individual components of the system, but they also create a wedge between ownership and decision-making. The classical intuition on the role of pooling risk in raising welfare is valid when ownership is evenly dispersed. However, when the ownership of some agents is concentrated in the hands of a few others, greater integration and diversification can lead to excessive risk taking and volatility and result in lower welfare. We also show that individuals undertake too little (too much) risk relative to the first best if the network is homogeneous (heterogeneous), and study optimal networks.

Financial Linkages, Portfolio Choice and Systemic Risk

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Abstract

Financial linkages smooth the shocks faced by individual components of the system, but they also create a wedge between ownership and decision-making. The classical intuition on the role of pooling risk in raising welfare is valid when ownership is evenly dispersed. However, when the ownership of some agents is concentrated in the hands of a few others, greater integration and diversification can lead to excessive risk taking and volatility and result in lower welfare. We also show that individuals undertake too little (too much) risk relative to the first best if the network is homogeneous (heterogeneous), and study optimal networks.

1 Introduction

Financial networks are an important feature of the economy. They reflect cross-ownership across corporations, borrowing and lending among banks, international financial flows and norms of risk sharing. Financial linkages have the potential to smooth the shocks and uncertainties faced by individual components of the system. But they also create a wedge between ownership on the one hand and control and decision making on the other hand. We wish to understand if deeper financial integration reduces volatility and raises welfare, and more generally what are the properties of an ideal financial network.

To study these questions we develop a model in which a collection of agents, interconnected through financial obligations, make a portfolio choice decision. The network reflects the claims that each agent has on others: so a link from A to B specifies the claims of A on the value of B. Agents have mean-variance preferences. Every agent i can invest his endowment in a risk-free asset (with return r) or in a (distinct) risky asset i (with mean $\mu_i > r$ and variance σ_i^2).[1] The portfolio choices of agents and the network of cross-ownerships together define the distribution of individual payoffs. Our aim is to understand how the network shapes individual payoffs and and the volatility of the system as a whole.

We begin by deriving a summary measure that aggregates all direct and indirect claims induced by the financial cross-holdings: we refer to this as *ownership*. Thus every cross-holdings network induces a *ownership* adjacency matrix Γ. In this matrix, the entry γ_{AB} summarizes all the direct and indirect claims that A has on the economic returns of B. The entry γ_{AA} is referred to as *self-ownership* of agent A: it captures the extent to which A bears the wealth effects of his portfolio choice. Our first observation is that optimal investment in a risky asset is inversely related to self-ownership. It follows that, other things being equal, the expected value and variance are higher for agents with greater ownership of low self-ownership agents, as these are the high risk takers.

In the wake of the 2008 financial crises, much attention has focused on systemic risk. Following recent work in this area, we study the covariance of agents' values (Arlotto and Scarsini (2009), Brunnermeier (2010), Meyer and Strulovici (2012, 2013)). In the basic model, returns from risky assets are independent, but the cross-holdings network induces correlations across agents' economic returns. We find that the covariance between two agents is higher if they have greater overlap in their ownership portfolio, and, in particular, if they share

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[1]For example, an agent may be a bank that can invest in government bonds or finance local entrepreneurs' risky projects.

ownership of low self-ownership agents.

Equipped with these basic results, we turn to the effects of changes in networks. Networks with low self-ownership induce higher investments in risky assets and, therefore, exhibit a higher mean but also higher volatility in value. This means that, *a priori*, the welfare effects of changes in networks are unclear. Inspired by the literature on finance, we explore changes in networks using the concepts of *integration* and *diversification*. A network S is said to be more integrated than network S' if every link in S is weakly stronger and some are strictly stronger. A network S is more diversified than network S' if every agent in S has a more diversified profile of ownerships. Our analysis offers the following general insight: the effects of integration and diversification depend crucially on the topology of the network. We discuss the effects of integration here; similar considerations arise in the case of diversification.

A regular network is one in which all nodes have a similar cross-ownership pattern. We show in such networks an increase in integration leads to higher aggregate utilities. However, greater integration in heterogenous networks – where ownership patterns differ widely across agents – actually lowers aggregate utilities. In symmetric networks, self-ownership is bounded from below, and this, in turn, sets an upper bound on the level of risky investment and, hence, on the costs of volatility. In heterogenous networks by contrast integration can sharply lower the self-ownership of some agents: this raises their risky investment disproportionately. This, in turn, pushes up volatility for everyone and may lower aggregate utilities; in fact, it may decrease the utility of all agents. We illustrate this argument through an analysis of a class of networks that are empirically salient: the *core-periphery networks*.

Core-periphery networks consist of a core and a periphery group of nodes. Every node in the core is connected to all other core nodes. Every peripheral node is connected all nodes in the core. Figure 1 illustrates a core-periphery network. Inter-bank networks have a core-periphery structure, and empirical work suggests that this finding holds for different definitions of financial obligations and different levels of aggregations, see Soramaki et al. (2007), Martinez-Jaramillo et al (2014), Craig and von Peter (2014). Vitali et al. (2011) report that transnational corporations form a giant bow-tie structure and that a large portion of control flows to a small tightly-knit core of financial institutions. McKinsey Global Institute (2014) reports that the network of international financial flows has a core-periphery structure, with the core constituted of United States and Western Europe and the rest of the world comprising the periphery.

Our theoretical predictions on risk taking and volatility appear to be broadly consistent with recent empirical results on these networks. Our key result that core banks take more

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risk than periphery banks is consistent with recent empirical studies on inter-bank networks, e.g.,van Lelyveld and Veld (2012) and Langfield, Liu and Ota (2014). We show that there may be greater volatility and lower welfare with growing integration in core-periphery networks: this is consistent with the finding of Kose et al. (2006) that greater international integration sometimes increases volatility in consumption and incomes at the country level.

We then turn to a normative study of networks. Given a fixed network, we derive the socially optimal portfolio of investments. This characterization clarifies the externality generated by financial linkages: an agent focuses exclusively on his own risk exposure, whereas the collective optimum entails a trade-off between expected returns and the sum of own *and* others' variance. The general insight here is that an agent will take too much risk relative to what is collectively desirable when his ownership is concentrated in a few hands. By contrast, investment in risky assets is too low relative to what is collectively desirable when cross-ownerships are widely dispersed.

Finally, we study optimal network design. Deeper and more extensive ties smooth returns, but, by lowering self-ownership, they also raise investments in risky assets. This raises both expected returns and the variance. Our analysis of optimal networks clarifies that deepening the linkages in a symmetric way resolves the tension: the *first-best* network is a complete network in which every agent owns exactly $1/n$ of everyone else. This is also true when the planner is concerned about systemic risk. To complete the circle, we examine optimal networks in a setting where agents make portfolio decisions. We establish that the *second-best* network is also complete, but the strength of ties varies inversely with the weight placed on systemic risk in the planner's objective function.

Our paper builds on two important strands of research. The first line of work is the research on cross-holdings and linkages (Brioschi, Buzzacchi, and Colombo (1989), Eisenberg, and Noe (2001), Fedenia, Hodder, and Triantis (1994) and the recent work of Elliott, Golub abd Jackson (2014)). The second line of work is the distinction between ownership and control; here, we draw on the long and distinguished tradition that began with the classic work of Berle and Means (1932) and on the more recent work by Fama and Jensen (1983) and Shleifer and Vishnu (1989).[2] To the best of our knowledge, our paper is the first to study the implications of portfolio choice for financial integration and diversification and the optimal design of networks within a common framework.

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[2] "The property owner who invests in a modern corporation so far surrenders his wealth to those in control of the corporation that he has exchanged the position of independent owner for one in which he may become merely recipient of the wages of capital" (Berle and Means (1932), page 355).

We conclude with a discussion of the recent work on contagion in financial networks. The role of linkages in transmitting and possibly amplifying shocks across components of the financial system has attracted much attention. Early contributions include Allen and Gale (2000); recent work includes Acemoglu, Ozdagler and Talbrezi (2015), Cabrales, Gottardi and Vega-Redondo (2011), Elliott, Golub and Jackson (2014), Elliott and Hazell (2015), Greenwood, Landier, and Thesmar (2015) and Gai and Kapadia (2010). For a survey, see Cabrales, Gale and Gottardi (2015). In this work, the focus is on how the network of linkages mediates the spread of exogenous shocks across individual components of the system and how the nature of the optimal network varies with the size distribution of shocks. The distinguishing feature of our work is that the origin of the shocks – the investments in risky assets – is itself an object of individual decision making. Thus, the focus of our work is, first, on how the network of linkages shapes the level of risk taking by agents and, second, on how it spreads the rewards of the risky choices across different parts of the system. Therefore, our work on the effects of integration and diversification and on optimal network design should be seen as complementary to the existing body of work.[3]

Section 2 introduces the model. Section 3 presents our characterization of risk taking in a network, and Section 4 studies the effects of changes in networks on welfare. Section 5 studies first-best investments and Section 6 examines optimal networks. Section 7 comments on different aspects of the model and discusses a number of extensions. Here, we provide a fairly detailed analysis of one, alternative models of agency and the separation between ownership and control, two, correlations across returns of risky assets, and three, endogenous networks. We also relate our work to the literature on informal risk-sharing in developing countries. The proofs of the results are presented in the Appendix. An online Appendix contains supplementary material that covers the technical results in Section 7.

2   Model

There are $\mathcal{N} = \{1, ..., n\}$, $n \geq 2$ agents. Agent i has has an endowment $w_i \in \mathbb{R}$ and chooses to allocate it between a safe asset, with return $r > 0$, and a (personal) risky project i, with return z_i. We assume that z_i is normally distributed with mean $\mu_i > r$ and variance σ_i^2.

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[3]In particular, our definitions of integration and diversification have a similar motivation as in Elliott, Golub and Jackson (2014), but we study the effects on utility and covariance of values.

For simplicity, in the basic model, we assume that the n risky projects are uncorrelated.[4] Investments by agent i in the risky asset and the safe asset are denoted by $\beta_i \in [0, w_i]$ and $\omega_i - \beta_i$, respectively. Let $\beta = \{\beta_1, ..., \beta_n\}$ denote the profile of investments.

Agents are embedded in a network of cross-holdings; we represent the network as an $n \times n$ matrix S, with $s_{ii} = 0$, $s_{ij} \geq 0$ and $\sum_{j \in \mathcal{N}} s_{ji} < 1$ for all $i \in \mathcal{N}$. We interpret s_{ij} as the claim that agent i has on agent j's economic value V_j.

Let D be a $n \times n$ diagonal matrix, in which the ith diagonal element is $1 - \sum_{j \in \mathcal{N}} s_{ji}$. Define $\Gamma = D[I - S]^{-1}$. Observe that since for every $i \in \mathcal{N}$, $\sum_{j \in \mathcal{N}} s_{ji} < 1$, it follows that we can write $\Gamma = D \sum_{k=0}^{\infty} S^k$. Therefore, the γ_{ij} cell is obtained by summing up all weighted paths from i to j in the cross-holdings network S– i.e., for every $i \neq j$,

$$\gamma_{ij} = \left[1 - \sum_{j \in \mathcal{N}} s_{ji} \right] \left[0 + s_{ij} + \sum_k s_{ik} s_{kj} + .. \right].$$

It is then natural to interpret γ_{ij} as i's *ownership* of j. Finally, note that Γ is column-stochastic, $\gamma_{ii} = 1 - \sum_{j \neq i} \gamma_{ji}$. We borrow the formulation of cross-holdings from Brioschi, Buzzacchi, and Colombo (1989), Fedenia, Hodder, and Triantis (1994), and, more recently, Elliott, Golub and Jackson (2014) and Elliott and Hazell (2015). Following Elliott, Golub and Jackson (2014) we interpret this formulation as a linear approximation of an underlying set of contracts linking financial institutions.[5]

Empirical work has highlighted the prominence of a core-periphery structure in financial networks (Bech and Atalay (2010), Afonso and Lagos (2012), McKinsey Global Institute (2014), Van Lelyveld I., and t' Veld (2012)). We present the ownership matrix for this network.

Example 1 *Ownership in the core-periphery network*

There are n_p peripheral agents and n_c central agents, $n_p + n_c = n$; i_c and i_p refer to the (generic) central and peripheral agent. A link between two central agents has strength $s_{i_c j_c} = s$, and a link between a central and a peripheral agent has strength $s_{i_c i_p} = s_{i_p i_c} = \hat{s}$, and there are no other links. Figure 1 presents such a network.

Computations presented in the Appendix show that the self-ownership of a central node

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[4]The online Appendix presents our analysis of assets with correlated returns. See, also, the discussion in Section 7.

[5]For every S, we can obtain a corresponding Γ; however, the converse is not always the case. For sufficient conditions on Γ that guarantee the existence of corresponding S, see Elliot, Golub and Jackson (2014).
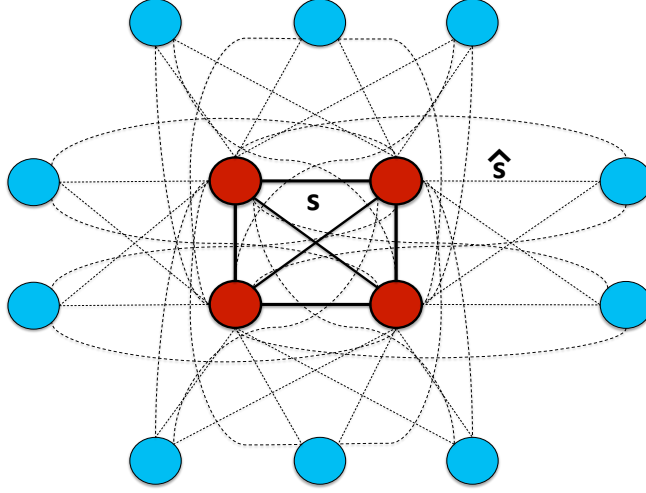
Figure 1: Core-periphery Network, $n_c = 4$, $n_p = 10$.

i_c and a peripheral node are, respectively,

$$\gamma_{i_c,i_c} = \frac{[1 - (n_c - 1)s - n_p\hat{s}][1 - (n_c - 2)s - n_c n_p \hat{s}^2 + n_p \hat{s}^2]}{(s+1)[1 - s(n_c - 1) - n_c n_p \hat{s}^2]},$$

$$\gamma_{i_p,i_p} = \frac{[1 - n_c\hat{s}][1 - (n_c - 1)s - n_c\hat{s}^2(n_p - 1)]}{1 - s(n_c - 1) - n_c n_p \hat{s}^2}.$$

Similarly, the cross-ownerships are given by:

$$\gamma_{i_c,j_p} = \frac{[1 - (n_c - 1)s - n_p\hat{s}]\hat{s}}{1 - s(n_c - 1) - n_c n_p \hat{s}^2} \quad \text{and} \quad \gamma_{j_p,i_c} = \frac{[1 - n_c\hat{s}]\hat{s}}{1 - s(n_c - 1) - n_c n_p \hat{s}^2},$$

$$\gamma_{i_c,j_c} = \frac{[1 - (n_c - 1)s - n_p\hat{s}][s + n_p\hat{s}^2]}{(s+1)(1 - s(n_c - 1) - n_c n_p \hat{s}^2)} \quad \text{and} \quad \gamma_{i_p,j_p} = \frac{[1 - n_c\hat{s}]n_c\hat{s}^2}{1 - s(n_c - 1) - n_c n_p \hat{s}^2}.$$

We note that the complete network ($n_p = 0$) and the star network ($n_p = n - 1$) constitute special cases of the core-periphery network. $\qquad\square$

We now define the economic value V_i. For a realization z_i and agent i's investments $(\beta_i, w_i - \beta_i)$, the returns generated by i are given by

$$W_i = \beta_i z_i + (w_i - \beta_i)r. \tag{1}$$

It then follows that the economic value of agent i is

$$V_i = \sum_j \gamma_{ij} W_j. \tag{2}$$

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We now turn to the choice problem for agents. We assume that agents seek to maximize a mean-variance utility function:[6]

$$U_i(\beta_i, \beta_{-i}) = E[V_i(\beta)] - \frac{\alpha}{2} Var[V_i(\beta)].$$

Using expressions (1) and (2), we can rewrite expected utility as

$$U_i(\beta_i, \beta_{-i}) = \sum_{j \in \mathcal{N}} \gamma_{ij}[w_j r + \beta_j(\mu_j - r)] - \frac{\alpha}{2} \sum_{j \in \mathcal{N}} \gamma_{ij}^2 \beta_j^2 \sigma_j^2. \tag{3}$$

Let $\beta^* = (\beta_1, .., \beta_n)$ denote the vector of optimal choices.

We are especially interested in the effects of the network of cross-holdings on systemic risk. The network S and portfolio choice $\beta^*(S)$ define the economic values $\mathcal{V}(S) = \{V_1(S), ..., V_n(S)\}$. In the literature on systemic risk, we distinguish two strands of work. There is the theoretical work on supermodular stochastic ordering (SSP). A vector of random variables X dominates another vector Y according to SSP if, and only if, $E[F(X)] > E[F(Y)]$ for all supermodular functions F (Meyer and Strulovici (2012, 2013) and Arlotto and Scarsini (2009)). This is a natural and very elegant formulation, but it yields partial orderings in cases of interest. The macro finance literature has, consequently, tended to work with simpler (and more special-ized) measures of systemic risk, such as CoVar and systemic expected shortfall (Brunnermeier (2010), Acharya et al. (2010)). Both CoVar and the systemic expected shortfall capture co-movements in the tails of random variables and yield a more complete ordering. Building on these observations, in this paper, we shall say that S exhibits greater systemic risk than S' whenever $\sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N} \setminus \{i\}} Cov(V_i(S), V_j(S)) > \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N} \setminus \{i\}} Cov(V_i(S'), V_j(S'))$.

3   Risk-taking in networks

We begin by characterizing optimal agent investments and then elaborate on the implications for utility and systemic risk.

Agents' utility is given by (3); observe that the cross partial derivatives with respect to investments are zero.[7] So, the optimal investment by agent i may be written as:

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[6] For a discussion of the foundations of mean-variance utility, see Gollier (2001).

[7] This means that agents' investment choices can be studied independently; this independence sets our paper apart from the literature on network games, which has been recently reviewed by Bramoulle and Kranton (2015) and Jackson and Zenou (2015).

$$\beta_i^* = \arg \max_{\beta_i \in [0, w_i]} \gamma_{ii}[w_i r + \beta_i(\mu_i - r)] - \frac{\alpha}{2}\gamma_{ii}^2\beta_i^2\sigma_i^2.$$

If agent i has no cross-holdings –i.e., $s_{ij} = s_{ji} = 0$ for all $i \neq j \in \mathcal{N}$– then $\gamma_{ii} = 1$, and, therefore, the optimal investment is

$$\hat{\beta}_i = \frac{\mu_i - r}{\alpha\sigma_i^2}.$$

We shall refer to $\hat{\beta}_i$ as agent i's *autarchy* investment. With this definition in place, we state our characterization result on optimal risk taking.

Proposition 1 *The optimal investment of agent i is:*

$$\beta_i^* = \min\left\{w_i, \frac{\hat{\beta}_i}{\gamma_{ii}}\right\}. \tag{4}$$

In an interior solution, expected value and variance for agent i are:

$$E[V_i] = r\sum_{j\in\mathcal{N}}\gamma_{ij}w_j + \sum_{j\in\mathcal{N}}\hat{\beta}_j(\mu_j - r)\frac{\gamma_{ij}}{\gamma_{jj}} \qquad Var[V_i] = \sum_{j\in\mathcal{N}}\hat{\beta}_j^2\sigma_j^2\left(\frac{\gamma_{ij}}{\gamma_{jj}}\right)^2, \tag{5}$$

and the covariance between V_i and V_j is:

$$Cov(V_i, V_j) = \sum_{l\in\mathcal{N}}\hat{\beta}_l^2\sigma_l^2\frac{\gamma_{il}\gamma_{jl}}{\gamma_{ll}^2}. \tag{6}$$

Proof: Suppose that the solution is interior. As the objective function is concave, the first-order condition is sufficient. Taking derivatives in (3) with respect to β_i and setting it equal to 0, immediately yields the required expression for optimal investments. Substituting the optimal investments in the expressions for the expected value and variance yields the expressions in the statement of the result. We now derive the expression for covariance between V_i and V_j. The covariance of the earnings of agent i and agent j is

$$
\begin{aligned}
Cov(V_i, V_j) &= \sum_{l\in\mathcal{N}}\gamma_{il}\beta_l\sum_{l'\in\mathcal{N}}\gamma_{jl'}\beta_{l'}\left(E[z_l z_{l'}] - \mu^2\right), \\
&= \sum_{l\in\mathcal{N}}\gamma_{il}\gamma_{jl}\beta_l^2\sigma_l^2\sigma_l^2,
\end{aligned}
$$

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where the second equality follows from the assumption that projects are independent. Using optimal investment (4), expression(6) follows. ∎

Proposition 1 yields a number of insights about the network determinants of risk taking. First, relative to autarky, cross-holdings raise agents' propensity to take risk: agent i's risk-taking investment is negatively related to his self-ownership, as captured by γ_{ii}. Thus, if two agents face similarly risky projects, $\mu_i = \mu_j$ and $\sigma_i^2 = \sigma_j^2$, then agent i invests more than agent j in the risky project if, and only if, $\gamma_{ii} < \gamma_{jj}$. This result follows from the agency problem that cross-holding networks generate: agent i optimizes the mean-variance utility of $\gamma_{ii} W_i$, and not of W_i.

The simplicity of optimal investment policy allows us to develop a relationship between networks and expected returns, volatility and correlations across agents' economic values. An inspection of the expressions $E[V_i]$ and $Var[V_i]$ reveals that agents with higher volatility and higher expected value are those with higher ownership of agents with low self-ownership, as the latter invest more in their risky project. For example, if $\sigma_i^2 = \sigma_j^2$, and $\mu_i = \mu_j$, for all i, j, then the variance of V_i is higher than the variance of V_j if, and only if,

$$\sum_{l \in \mathcal{N}} \left(\frac{\gamma_{il} - \gamma_{jl}}{\gamma_{ll}} \right)^2 > 0.$$

Following our discussion on systemic risk in Section 2, we now examine the covariance between the values of different agents. Proposition 1 tells us that the covariance between V_i and V_j is higher when agents i and j have a similar ownership structures and when they have common ownership of agents with low self-ownership, as summarized by:

$$\sum_{l \in \mathcal{N}} \frac{\gamma_{il} \gamma_{jl}}{\gamma_{ll}^2}.$$

We illustrate Proposition 1 through a detailed study of two stylized networks: the complete network and the star network. These two networks have been picked as they reflect the idea of homogeneity (or regularity) and heterogeneity in a stark form. Our interest is in the nature of the network and so, for expositional simplicity, we assume that all agents are ex-ante identical–i.e., $\sigma_i^2 = \sigma^2$, $\mu_i = \mu$ and $w_i = w$ for all i.

Example 2 *The Complete Network*

In a complete network, every (ordered) pair of agents has a directed link of strength s. Figure 2 illustrates changing ownership within a complete network topology.
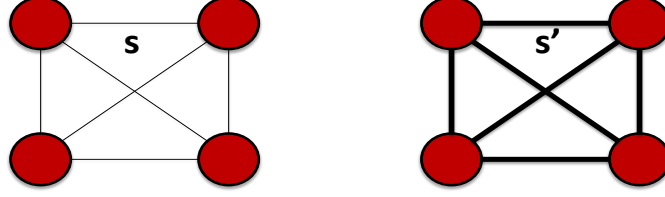
Figure 2: Changing ownership in a complete network

From Example 1 we can deduce, by setting $n_p = 0$, that the ownership matrix Γ in a complete network is

$$\gamma_{ij} = \frac{s}{s+1} \quad \text{and} \quad \gamma_{ii} = 1 - (n-1)\gamma_{ij}.$$

Greater s lowers self-ownership and, from Proposition 1, we know that this means that all agents raise their investment in risky assets. As a consequence, both the expected value $E[V_i]$ and the variance $Var[V_i]$ increase in s. Substituting the ownerships in expression (3) tells us that the expected utility of each agent is increasing in s. Furthermore, the covariance of agent i's value and agent j's value is

$$Cov(V_i, V_j) = \frac{(\mu - r)^2}{\alpha^2 \sigma^2} \frac{s[2 - s(n-2)]}{(1 + 2s - n)^2},$$

which is increasing and convex in s. It follows that systemic risk in a complete network–i.e., $\sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N} \setminus \{i\}} Cov(V_i, V_j)$– is also increasing and convex in the strength of each connection. The details of these computations are provided in the Appendix.[8] $\square$

Example 3 *The Star Network*

In Example 1, we get the star network if we set $n_c = 1$. Figure 3 illustrates changing ownership within a star network topology.

The self-ownerships of central and peripheral agents are, respectively:

$$\gamma_{i_c i_c} = \frac{1 - n_p \hat{s}}{1 - n_p \hat{s}^2} \quad \text{and} \quad \gamma_{i_p i_p} = \frac{[1 - \hat{s}][1 - \hat{s}^2(n_p - 1)]}{1 - n_p \hat{s}^2}.$$

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[8]The implications for risk taking and expected utility obtained here extend to a multi-sectoral generalization of the complete network. In a multi-sector network the within-sector links are stronger than the cross-sector links.
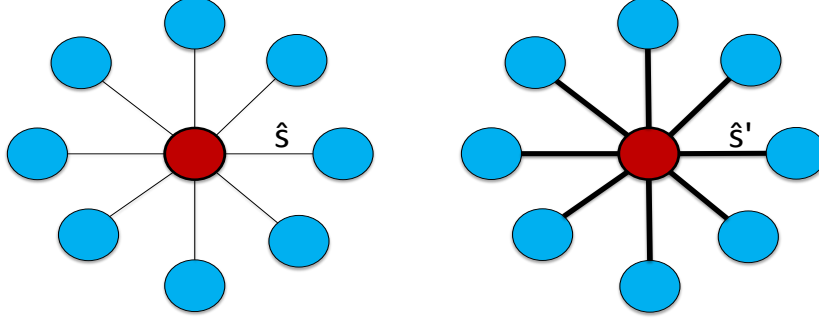
Figure 3: Changing ownership in a star

It is possible to verify that $\gamma_{i_c i_c} < \gamma_{i_p i_p}$ and, from Proposition 1, this implies that the central agent makes larger investments in the risky asset. We now turn to the expected value and volatility.

The cross-ownerships in the star network are, respectively:

$$\gamma_{i_c,j_p} = \frac{[1 - n_p \hat{s}]\hat{s}}{1 - n_p \hat{s}^2}, \quad \gamma_{j_p,i_c} = \frac{[1 - \hat{s}]\hat{s}}{1 - n_p \hat{s}^2}, \quad \gamma_{i_p,j_p} = \frac{[1 - \hat{s}]\hat{s}^2}{1 - n_p \hat{s}^2}.$$

Substituting these expressions in the mean and variance equations in (5) reveals that for small $\hat{s}$, the central player has a higher mean and higher variance than the peripheral players; the converse holds for large $\hat{s}$. The intuition is the following. For small $\hat{s}$, each agent has high self-ownership, and, therefore, his own investment is what matters more for the mean and variance of his own value. Since the central agent invests more in the risky asset relative to the peripheral players, he obtains higher returns but is also exposed to higher variance. When $\hat{s}$ is high, the central player has very little self-ownership and very little ownership of peripheral players. In contrast, the peripheral players have positive and large ownership of the central player. As a consequence, peripheral players absorb the large risky investments that the central player undertakes.

These effects have implications for individual utility and for systemic risk. When $\hat{s}$ is low, the utility of the central player is higher than the utility of the peripheral players because the center has a higher mean than the peripheral players and the cost of the variance is low, as each agent invests moderately in the risky asset. When $\hat{s}$ is high, the central player is better off again; in this case, he has a lower mean than the peripheral players, who, however, face very high variance. For intermediate values of $\hat{s}$, the peripheral players may be better off than the center.

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Finally, in a star network, the covariance between the center and a peripheral agent, and the covariance between two peripheral agents are, respectively,

$$
\begin{aligned}
Cov(V_{i_c}, V_{i_p}) &= \frac{(\mu - r)^2}{\alpha^2 \sigma^2} \hat{s} \left[\frac{1 - \hat{s}}{1 - n_p \hat{s}} + \frac{(1 - n_p \hat{s})(1 - (n_p - 1)\hat{s}^3)}{(1 - \hat{s})(1 - (n_p - 1)\hat{s}^2)} \right] \\
Cov(V_{i_p}, V_{j_p}) &= \frac{(\mu - r)^2}{\alpha^2 \sigma^2} \hat{s}^2 \left[\frac{2}{1 - \hat{s}^2(n_p - 1)} + \frac{(1 - \hat{s})^2}{(1 - n_p \hat{s})^2} + \frac{(n_p - 2)(1 - \hat{s})\hat{s}^2}{(1 - (n_p - 1)\hat{s}^2)^2} \right].
\end{aligned}
$$

When $\hat{s}$ is low, the returns of a central agent and a peripheral agent are more correlated than the returns of two peripheral agents. However, when $\hat{s}$ is large, the returns to peripheral agents depend greatly on the central agent's performance, and, therefore, $Cov(V_{i_p}, V_{j_p}) > Cov(V_{i_c}, V_{i_p})$. The details of these derivations are presented in the Appendix. $\square$

These examples illustrate the powerful role of the network in shaping portfolio choice; in particular, they suggest that stronger ties have very different effects on risk taking and on systemic risk, depending on the architecture of the underlying network. We examine this point more systematically in the next section.

4 Integration and Diversification

Empirical research in macroeconomics and in finance shows that financial linkages have deepened over the past three decades–e.g., Kose, Prasad, Rogoff and Wei (2006), Lane and Milesi-Ferretti (2003). Motivated by this work, we will study two types of changes in networks: integration and diversification.

For expositional simplicity, in this section, we assume that agents are ex-ante identical–i.e., $\mu_i = \mu$, $\sigma_i^2 = \sigma^2$ and $w_i = w$. We start by developing some general results on how changes in networks affect aggregate welfare.

Recall that an agent's investment in autarchy is given by $\hat{\beta} = (\mu - r)/\alpha \sigma^2$. We can write agent utility under optimal investment as

$$
U_i(S) = wr \sum_j \gamma_{ij} + \hat{\beta}(\mu - r) \sum_j \left[\frac{\gamma_{ij}}{\gamma_{jj}} - \frac{1}{2} \frac{\gamma_{ij}^2}{\gamma_{jj}^2} \right].
$$

Aggregate welfare is the sum of agent utilities:

$$
\begin{aligned}
W(S) &= \sum_i \left[wr \sum_j \gamma_{ij} + \hat{\beta}(\mu - r) \sum_j \left[\frac{\gamma_{ij}}{\gamma_{jj}} - \frac{1}{2} \frac{\gamma_{ij}^2}{\gamma_{jj}^2} \right] \right] \\
&= wrn + \frac{(\mu - r)^2}{\alpha \sigma^2} \sum_i \sum_j \left[\frac{\gamma_{ij}}{\gamma_{jj}} - \frac{1}{2} \frac{\gamma_{ij}^2}{\gamma_{jj}^2} \right],
\end{aligned}
$$

where we have used the fact that Γ is column stochastic. We then obtain that $W(S) > W(S')$, if, and only if,

$$
\sum_{j \in \mathcal{N}} \left[\frac{1}{\gamma_{jj}(S)} - \frac{1}{\gamma_{jj}(S')} \right] > \frac{1}{2} \sum_j \sum_{i \in \mathcal{N}} \left[\frac{\gamma_{ij}^2(S)}{\gamma_{jj}^2(S)} - \frac{\gamma_{ij}^2(S')}{\gamma_{jj}^2(S')} \right]. \tag{7}
$$

The expression $\sum_{j \in \mathcal{N}} 1/\gamma_{jj}$ may be seen as a measure of the aggregate level of risk taking in the network. It is proportional to the aggregate expected returns generated by a network: low self-ownership generates high aggregate expected returns. The term

$$
\sum_{j \in \mathcal{N}} \frac{\gamma_{ij}^2(S)}{\gamma_{jj}^2(S)}
$$

reflects the costs of aggregate volatility. The inequality expresses the costs and benefits of changes in expected returns vis-a-vis changes in variance in terms of the ownership matrix Γ.[9] We will apply this inequality to compare welfare across different cross-holding networks.

In general, the relation between S and Γ can be quite complicated; to make progress, we focus on first- and second-order effects of changes in the linkages. For this approximation to be informative, we need to assume that the strength of each link in S is sufficiently small–i.e., the network of cross-holdings S is *thin*. Define $\eta_i^{in} = \sum_{j \in \mathcal{N}} s_{ji}$ and $\eta_i^{out} = \sum_{j \in \mathcal{N}} s_{ij}$ as the in-degree and out-degree of $i \in \mathcal{N}$, respectively. For thin networks, we can then write the terms in Γ as:

$$
\gamma_{ii} \simeq 1 - \eta_i^{in} + \sum_l s_{il} s_{li} \text{ and } \gamma_{ij} \simeq s_{ij}(1 - \eta_i^{in}) + \sum_l s_{il} s_{lj}. \tag{8}
$$

---

[9]To get a sense of how the network structure affects the two sides of the inequality, consider two scenarios: 1) where j's ownership is evenly distributed $\gamma_{ij} = 1/n$ for all $i \in \mathcal{N}$; and 2) where $\gamma'_{jj} = 1/n$, and the remaining ownership of every j is concentrated in the hands of a single agent 1, $\gamma'_{1j} = (n-1)/n$ and $\gamma'_{ij} = 0$ for all $i \neq 1, j$. Inequality (7) tells us that the left-hand side is 0, but the right-hand side is higher by a factor $n(n-1)$.
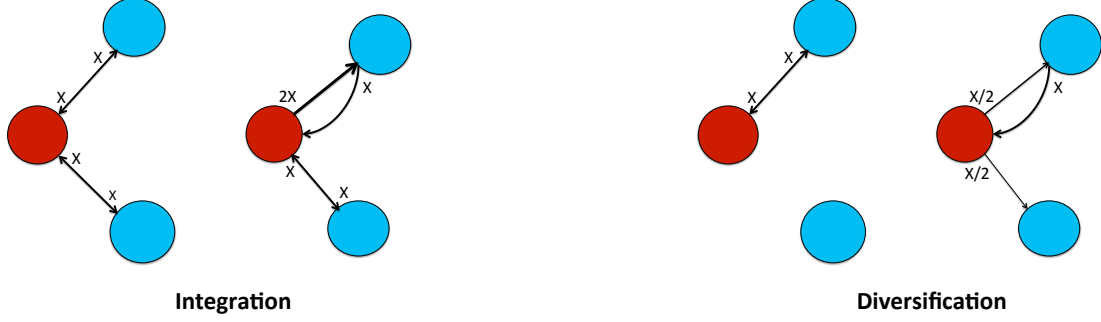
Figure 4: Changes in Networks

This, in turn, implies that

$$\frac{\gamma_{ii}}{\gamma_{ij}} \simeq s_{ij} + \sum_l s_{il} s_{lj} \text{ and } \frac{\gamma_{ii}^2}{\gamma_{ij}^2} \simeq s_{ij}^2. \tag{9}$$

With these simplifications in hand, we are ready to state our first result on thin networks.

Proposition 2 *Assume that* $\sigma_i^2 = \sigma^2$, $\mu_i = \mu$ *and* $w_i = w$. *There exist* $\overline{w} > 0$ *and* $\bar{s} > 0$ *so that if* $w > \overline{w}$ *and* $||S||_{\max} < \bar{s}$ *and* $||S'||_{\max} < \bar{s}$, *then* $W(S) > W(S')$ *if, and only if,*

$$\sum_{i \in \mathcal{N}} \left[\eta_i^{out}(S)(1 + \eta_i^{in}(S)) - \eta_i^{out}(S')(1 + \eta_i^{in}(S')) \right] \geq \frac{1}{2} \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N}} \left[s_{ij}^2 - s_{ij}'^2 \right]. \tag{10}$$

The inequality in the Proposition follows from substituting the ratios (9) in equation (7), and rearranging terms.

We now formally define integration and diversification in networks. For a vector $s_i = \{s_{i1}, ..., s_{in}\}$, define the variance of s_i as $\sigma_{s_i}^2 = \sum_j (s_{ij} - \eta_i^{out}/(n-1))^2$.

Definition 1 *Integration* *We say that* S *is more integrated than* S' *if* $s_{ij} \geq s_{ij}'$ $\forall i, j \in \mathcal{N}$, *and* $s_{ij} > s_{ij}'$ *for some* $i, j \in \mathcal{N}$.

The definition of integration reflects the idea that links between entities have become stronger.

Definition 2 *Diversification* *We say that* S' *is more diversified than* S *if* $\eta_i^{out}(S') = \eta_i^{out}(S)$ *and* $\sigma_{s_i'}^2 \leq \sigma_{s_i}^2$ $\forall i \in \mathcal{N}$, *and* $\sigma_{s_i'}^2 < \sigma_{s_i}^2$ *for some* $i \in \mathcal{N}$.

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The definition of diversification reflects the idea that an existing sum of strength of ties is more evenly spread out. Figure 4 illustrates these definitions.

Our next result builds on Proposition 2 to draw out the effects of greater integration and diversification on aggregate utility.

Corollary 1 *Assume that $\sigma_i^2 = \sigma^2$, $\mu_i = \mu$ and $w_i = w$. There exist $\overline{w} > 0$ and $\bar{s} > 0$ so that if $w > \overline{w}$ and $||S||_{\max} < \bar{s}$ and $||S'||_{\max} < \bar{s}$, the following holds:*

1. *If S is more integrated than S', then $W(S) > W(S')$.*

2. *If S is more diversified than S', then $W(S) > W(S')$ if, and only if,*

$$2 \sum_i \eta_i^{out}(S') \left[\eta_i^{in}(S') - \eta_i^{in}(S) \right] < \sum_i [\sigma_{s_i'}^2 - \sigma_{s_i}^2]. \tag{11}$$

An increase in integration lowers self-ownership and pushes up investment in risky assets. Higher investment in risky assets, in turn, raises expected returns and variability in returns. However, the costs of the increased variability are second-order and, in thin networks, are dominated by the benefits of higher expected returns.[10]

We now take up diversification: consider the case of a network in which some agents have high in-degree and some agents have low in-degree. In a thin network, the former have low self-ownership and, therefore, make large risky investments; by contrast, the latter group of agents have a high self-ownership and invest less in the risky asset. An increase in diversification leads to a reallocation away from high in-degree nodes to low in-degree nodes. Since investment in the risky project is proportional to $1/\gamma_{ii}$, high in-degree agents lower their risky investment more than the low in-degree agents raise it. Hence, both aggregate volatility and expected returns decline. Condition (11) in Proposition 1 clarifies the relative magnitude of these changes. In particular, the right-hand side reflects the decrease in the cost of the variance due to diversification; since S is more diversified than S', the right-hand side is always positive. The left-hand side represents the change in the expected aggregate returns due to diversification.

We illustrate the trade-offs involved in diversification with the help of two simple examples.

Example 4 *Diversification, heterogeneous networks and welfare.*

---

[10]The effects of integration on agent utilities will vary: if A owns a much larger part of B, then the ownership and, hence, the utility of B will typically go down, while the utility of A will go up.

Suppose $n = 4$ and network S is defined as: $s'_{12} = s'_{43} = \epsilon$ and $s'_{13} = s'_{42} = 2\epsilon$, and all other links are zero. Next, define network S as: $s_{12} = s_{43} = s_{13} = s_{42} = 3\epsilon/2$, and all other links are zero. Note that S is more diversified than S', but the in-degree of each agent is the same in S and in S'. Let ϵ be small so that the network is thin. Thus, the left-hand side of condition (11) equals zero, and the right-hand side is positive. It follows that aggregate welfare is higher under the more diversified network S.

Suppose that $n = 3$, and network S' is defined as follows: $s'_{12} = s'_{21} = \epsilon$ and all other links are 0. The network S is defined as $s_{12} = s_{13} = \epsilon/2$, and $s_{21} = s'_{21}$. This is a thin network for sufficiently small ϵ. Note that the left-hand side of condition (11) is equal to ϵ^2, and the right-hand side is equal to $\epsilon^2/2$. Aggregate welfare is lower in the more diversified network S.
□

We have shown that in thin networks, deeper integration always increases aggregate utilities. We now turn to general networks. The discussion in Example 2 shows that if the network is complete, then deeper integration raises aggregate welfare, even if the network is not thin. We conclude this section by showing that aggregate welfare may decline with integration in asymmetric networks.

Recall that in a core periphery network, there are n_p peripheral agents and n_c core agents; following the empirical findings on inter-bank networks, we assume that $n_p >> n_c$. We increase integration by strengthening the ties between core and periphery agents–i.e, we increase $\hat{s}$. We note that international financial flows have grown dramatically over the past two decades. In particular, the ties between peripheral countries and core countries have become stronger (McKinsey Global Institute (2014)). The example we present here is very stylized but it reflects this broad empirical trend. Figure 3 reports one set of numerical analysis.

An increase in cross-ownerships between center and peripheral agents leads to a decrease in the self-ownership of core agents, who, therefore, take greater risk. When connections are strong, core agents do not internalize the large costs of the variance that their risky investments generate. Eventually, welfare, and in fact the utility of each agent, declines with greater integration. Finally, note that systemic risk also increases sharply with greater integration (broadly similar effects are seen if we strengthen ties between core nodes).

It is worth emphasizing that the above patterns are obtained in a setting where the nodes are ex-ante identical in terms of endowment and the links are symmetric. In empirically observed financial networks, the core nodes have larger endowments than the periphery nodes: this will further amplify the negative effects of integration on volatility of the system. On the
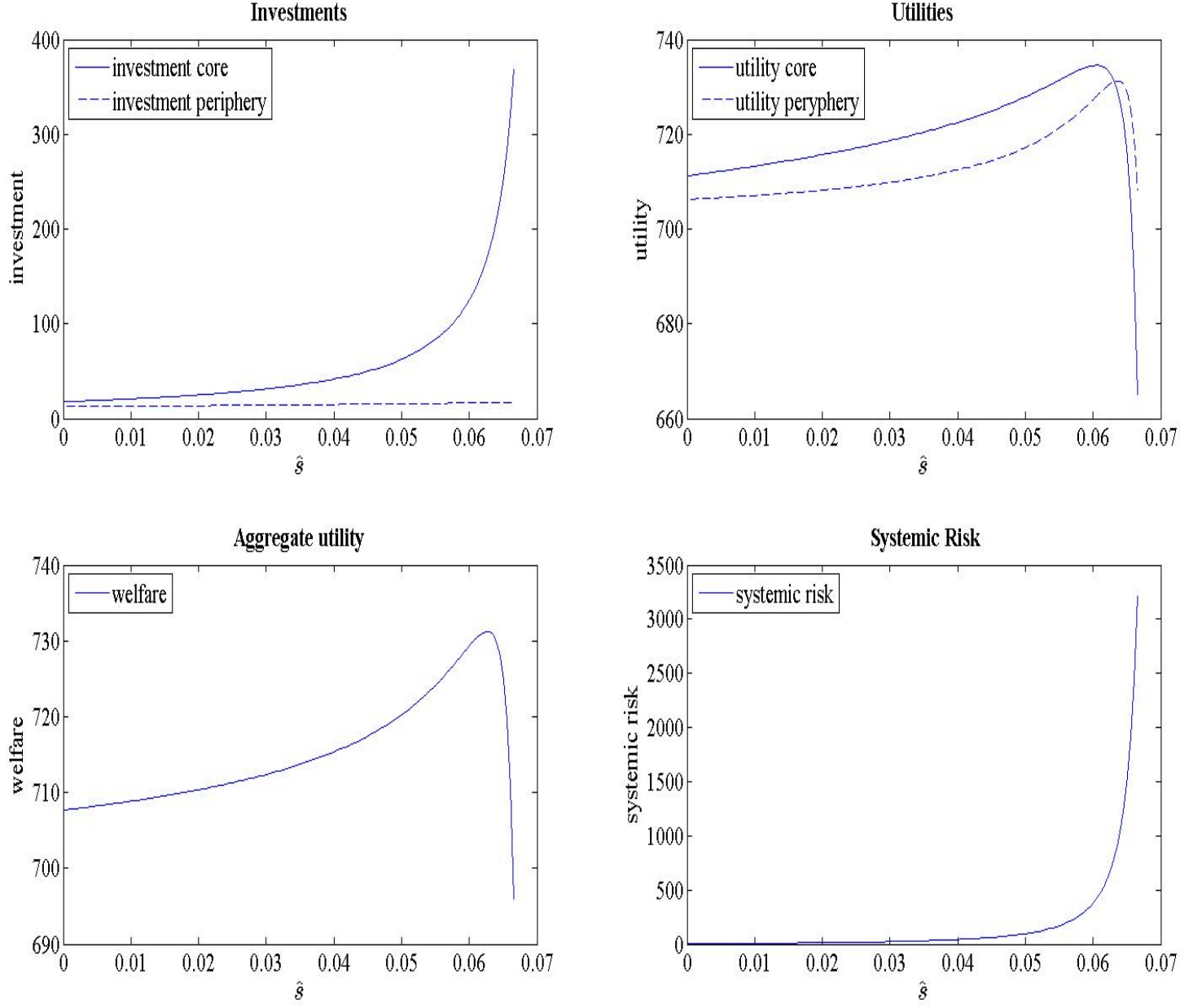
Figure 5: Integration in Core-Periphery Network; $n_c = 4$, $n_p = 10$, $\sigma = 0.4$, $\alpha = 0.5$, $\mu = 2$, $r = 1$, $w = 700$; $s = 0.1$, $\hat{s} = 0...0.065$.

issue of links, in some empirical contexts, such as international flows, the strength of the link from the periphery to the core has grown: this will further strengthen the decline in self-ownership of core nodes and amplify the effects we identify.[11]

The results in this section illustrate the powerful effects of network architecture on portfolio choice and welfare. They motivate a normative analysis of networks.

5   Optimal investments and the nature of externalities

This section presents a characterization of first-best investments in networks and then examines the difference between first-best and individually optimal investments. This leads us to study the costs of decentralization across networks.

We suppose that the 'planner' seeks to maximize an objective function that gives weight to agent utilities and also to systemic risk. The planner seeks to maximize

$$W^P(\beta, S) = \sum_{i \in \mathcal{N}} E[V_i] - \frac{\alpha}{2} \sum_{i \in \mathcal{N}} Var[V_i] - \frac{\phi}{2} \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N} \setminus \{i\}} Cov[V_i, V_j], \qquad (12)$$

where $\phi \geq 0$. When $\phi = 0$, the objective (12) is the objective of an utilitarian planner–i.e., $W(\beta, S) = \sum_i U_i(\beta, S)$. Furthermore, since $Var[V] = \sum_{i \in \mathcal{N}} Var[V_i] + \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N} \setminus \{i\}} Cov[V_i, V_j]$, we have that when $\phi = \alpha$, the social planner has mean-variance preferences with regard to aggregate output $V = \sum_i V_i$.

For a given S, the planner chooses investments in risky assets, $\beta^P = \{\beta_1^P, \beta_2^P, .., \beta_n^P\}$, to maximize (12). We obtain:

Proposition 3 *The optimal investment of the social planner in risky project $i = 1, ..., n$ is given by*

$$\beta_i^P = \min \left[w_i, \hat{\beta}_i \frac{\alpha}{\phi + (\alpha - \phi) \sum_{j \in \mathcal{N}} \gamma_{ji}^2} \right]. \qquad (13)$$

Observe that the optimal investment in risky project i decreases with the cost of systemic risk and the cost of volatility. In order to understand the externalities created by the network of holdings, we compare the marginal utility of increasing β_i for agent i, with the marginal

---

[11]A variety of financial networks have been empirically studied in recent years; see the introduction for references to this literature.

utility of the utilitarian planner. So, by setting $\phi = 0$, we have:

$$
\begin{aligned}
\frac{\partial U_i}{\partial \beta_i} &= (\mu_i - r)\gamma_{ii} - \alpha\sigma_i^2\beta_i\gamma_{ii}^2, \\
\frac{\partial W(S)}{\partial \beta_i} &= (\mu_i - r) - \alpha\sigma_i^2\beta_i\sum_{j\in\mathcal{N}}\gamma_{ji}^2.
\end{aligned}
$$

The agent ignores the impact of his risky investment on the aggregate expected returns *and* also on the sum of the agent's variance. In particular, an agent underestimates the impact of his investment on the aggregated expected value by $(1 - \gamma_{ii})$, and on the sum of variances by $\sum_{j\neq i}\gamma_{ji}^2$. Note that $\sum_{j\neq i}\gamma_{ji}^2$ is higher when the ownership of agent i is concentrated in a few other agents. This yields the following general insight: when the cross-holding network of agent i is highly concentrated, agent i's investment in risky assets is too high relative to what is collectively optimal. The converse is true if agent i's cross-holdings are widely dispersed.

Corollary 2 *Suppose that the planner is utilitarian–i.e., $\phi = 0$. Agent i over-invests as compared to the utilitarian planner, $\beta_i > \beta_i^P$ if, and only if,*

$$
\gamma_{ii} < \sum_{j\in\mathcal{N}}\gamma_{ji}^2.
$$

We now consider how the network affects the cost of decentralization. For simplicity, we focus on a utilitarian planner. Given a network S, the cost of decentralization is defined as

$$
K(S) = W(\beta^P, S) - W(\beta^*, S).
$$

Using Proposition 1 and Proposition 3, we obtain

$$
K(S) = \frac{(\mu - r)^2}{\alpha\sigma^2}\left[\sum_j\left(\frac{1}{\sum_l\gamma_{lj}^2} - \frac{1}{\gamma_{jj}}\right) - \frac{1}{2}\sum_i\sum_j\left(\gamma_{ij}^2\frac{1}{(\sum_l\gamma_{lj}^2)^2} - \frac{1}{\gamma_{jj}^2}\right)\right].
$$

We would like to order networks in terms of this cost of decentralization. While it is difficult to obtain a result when comparing arbitrary networks, we are able to make progress if we restrict attention to thin networks.

Proposition 4 *Assume that $\sigma_i^2 = \sigma^2$, $\mu_i = \mu$ and $w_i = w$ for all $i \in \mathcal{N}$. Suppose that S and S' are both thin networks and that the planner is utilitarian. There exist $\overline{w} > 0$ and $\bar{s} > 0$ so*

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that if $w > \overline{w}$ and $||S||_{\max} < \bar{s}$ and $||S'||_{\max} < \bar{s}$, the cost of decentralization is higher under S than under S' if, and only if,

$$\sum_j \eta_j^{in}(S)^2 > \sum_j \eta_j^{in}(S')^2. \tag{14}$$

Note that if S is more integrated than S', then $\eta_i^{in}(S) \geq \eta_i^{in}(S')$ for all $i \in \mathcal{N}$, and the inequality is strict for some i, which implies that condition (14) holds. That is, the cost of decentralization is higher in more-integrated networks. Intuitively, by increasing integration, agents' self-ownership decreases, and, therefore, the agency problem is stronger.

On the other hand, take two network S and S' for which the sum of in-degrees across agents is constant. Then, condition (14) tells us that the cost of decentralization is higher in networks in which in-degrees are concentrated on a few nodes, as in the core-periphery network. In these networks, it follows from Corollary 2, that the few agents with a large in-degree over-invest in the risky asset, creating far too much variability among the connected agents.

6   Optimal Network Design

This section considers the nature of the optimal network. It is useful to separately develop both a first-best and a second-best analysis. In the first-best analysis, the planner designs the network S to maximize objective (12) and dictates collectively optimal investments according to (13). In the second-best analysis, the planner designs the network S to maximize objective (12) but takes into account that, for a given S, agents choose investments according to (4). The following result summarizes our analysis.

Proposition 5 *Assume that w_i is large for all $i \in \mathcal{N}$.*

1. *The first-best network design is the complete network with maximum link strength $s_{ij} = 1/(n-1)$ for all $i \neq j$.*

2. *The second-best network design is the complete network with link strength*

$$s_{ij} = \frac{1}{n-1} \frac{\alpha - \phi}{\alpha}, \text{ for all } i \neq j.$$

To solve the first-best and second-best design problem, we first derive the optimal Γ, and then we derive the network S that induces the optimal Γ. We start by establishing that homogeneous networks – where links and weights are spread evenly across nodes - dominate heterogeneous networks. This is because agents are risk-averse, and concentrated and unequal ownership exacerbates the costs of variance. This leads to a preference for homogeneous networks: networks where, for every i, $\gamma_{ji} = \gamma_{j'i}$ for all $j, j' \neq i$.

In the first-best, within homogeneous networks, stronger links are better, as they allow for greater smoothing of shocks, and this is welfare-improving due to agents' risk aversion. In the second-best design problem, within homogeneous network, the designer has to choose between networks in which agents have high self-ownership (and, therefore, make large investments in the risky asset) *versus* low self-ownership (when they take little risk). When the social planner is utilitarian, $\phi = 0$, the optimal network is invariant: $s_{ij} = 1/n - 1$ for all i, j both in the first-best and the second-best case. On the other hand, if the social planner cares about correlation across agents, then the larger the weight placed on systemic risk, the greater the aversion to correlations in agents' values. Thus, the optimal network in the decentralized setting will be less integrated than the optimal network in the first-best scenario.

7   Remarks on model and extensions

We have developed a model in which the network of financial obligations mediates agents' risk taking behavior. The framework allows us to discuss the costs and benefits of greater integration and greater diversification and how they depend on the underlying network's characteristics. We conclude by commenting on a number of extensions. An online Appendix contains supplementary material that covers technical results in the following extensions.

7.1   Ownership and control

We briefly explore the relationship between ownership and control. In line with this literature, in the basic model, we have taken the view that ownership does not translate into control in a straightforward way. For expositional purposes, we assume that ownership and control are completely separate. We now discuss two different ways of bringing ownership more in line with decision rights.

Suppose that γ_{ij} signifies that agent i has control over γ_{ij} fraction of agent j's initial endowment w_j. One way of interpreting this control is to say that agent i can invest $\gamma_{ij}w_j$ in

the risk-free asset or in the risky project i. In this interpretation, $\gamma_{ij}w_j$ is a transfer from j to i that occurs before shocks are realized. Therefore, Γ redefines the agents' initial endowments. Since, under the mean-variance preferences, initial endowments do not influence risk taking (unless the solution is corner), the network plays no important role.

In an alternative scenario, suppose that ownership conveys control, but the control is 'local': agent i can invest $w\gamma_{ij}$ in the risk-free asset and in the risky project of agent j. The choice of agent i is, then, a vector of investments $\beta_i = \{\beta_{i1}, ..., \beta_{in}\}$, where β_{ij} is the investment in risky project j of endowment $w_{ij} = \gamma_{ij}w_j$, and $\beta_{ij} \in [0, \gamma_{ij}w_j]$. It is possible to show that, in this case, individually optimal investment levels are independent of the network, and agents' choices mimic those of a central planner with mean-variance preferences over aggregate returns $V = \sum_i V_i$.

These two examples illustrate that whenever ownership gives control in a "frictionless" way, the role of the network in shaping risk taking is uninteresting.

7.2   Correlations across projects returns

We assume that the returns of the projects are uncorrelated, and this assumption, taken together with the assumption of mean-variance utility, implies that agents' actions are strategically independent. Furthermore, given this assumption, any form of correlation across agents' economic value is driven by the architecture of the cross-holdings network. In this sense, the assumption that projects are uncorrelated allows us to isolate the effects of cross-holdings on risk-taking behavior and aggregate outcomes. Here, we briefly discuss the case of the correlated assets.

We consider the setting with arbitrary correlations across assets. We show existence and derive sufficient conditions for uniqueness of an interior equilibrium. We then show, via examples, that asymmetric networks may lead to over-investment in risky assets, as in the case of uncorrelated projects.

7.3   Exogenous vs. endogenous network

In the benchmark model, we take the network of financial linkages as given. Financial linkages take a variety of forms, ranging across overnight loans, cross-ownership, and investments in long-term bonds. So, as a first step, it is reasonable to study risk taking taking as given a long term structure of cross-ownerships. However, the strong results we obtain motivate us

to look more closely at the issue of endogenous networks.

We explore a simple model in which agents simultaneously demand shares of other firms, supply shares of their own firm, and decide how much risk to take. In this setting, an equilibrium is a network Γ, a price vector $p = \{p_1, ..., p_n\}$ that specifies the price p_i of each share of i, and a profile of investment β, such that each agent's decision is optimal, agents' expectations are rational, and the price clears the market.

We show that there is always an equilibrium outcome that replicates the outcome of an utilitarian social planner who optimally designs the network and chooses agents' investments. Agents' heterogeneity – in terms of initial endowment and in terms of expected returns and variance of their risky projects – translates into different investment decisions and different prices, but the equilibrium network is always symmetric. This suggests that frictions in link formation may be necessary in order to generate asymmetric networks (such as the core-periphery network).

7.4   Risk-sharing and Informal Insurance

At a general level, our analysis also has implications for informal risk-sharing in developing countries. This literature presumes that greater risk sharing leads to smoothing in consumption and higher welfare (Genicot and Ray 1993; Ray, 1998; Townsend 1994). By contrast, we show that greater risk sharing in heterogenous networks can lead to a decline in aggregate welfare when individuals have the choice of uncertain and risky crops or technologies. Recent empirical work suggests that rural risk sharing networks are heterogenous (Jackson, Rodriguez-Barraquer, and Tan (2012), Milan (2015)); our analysis highlights the interplay between networks and agency problem in such contexts and should motivate further research on risk taking in village economies.

In a recent paper, Belhaj and Deroian (2012) study risk taking by agents located within a network. There are two modeling differences: they assume positive correlation in returns to risky assets and bilateral output sharing with no spillovers in ownership. So, with independent assets, there are no network effects in their model. Our focus is on the effects of integration and diversification and the design of optimal networks (with weights on systemic risk). These issues are not addressed in their paper.

Appendix

Derivation of Γ matrix for core-periphery matrix: We first derive the Γ matrix for a core-periphery matrix, S. In a core-periphery network there are n_p peripheral individuals and n_c central individuals, $n_p + n_c = n$; i_c is a (generic) central individual and i_p is a peripheral individual. A link between two central individuals is $s_{i_c j_c} = s$ and a link between a central and a peripheral individual $s_{i_c i_p} = s_{i_p i_c} = \hat{s}$, and there are no other links.

Denote by $k^t(i_c, i_c)$ the element $[S^t]_{ii}$ where i is a core player, $k^t(i_p, i_p)$ the element $[S^t]_{ii}$ where i is a peripheral player, $k^t(i_c, j_c)$ the element $[S^t]_{ij}$ where i and j, $i \neq j$ are core players, $k^t(i_p, j_p)$ the element $[S^t]_{ij}$ where i and j, $i \neq j$ are peripheral players, $k^t(i_c, j_p)$ the element $[S^t]_{ij}$ where i is a core player and j is a peripheral player. It is easy to verify that for every $t \geq 1$ we have

$$
\begin{pmatrix} k^t(i_c, i_c) \\ k^t(i_c, j_c) \\ k^t(i_c, j_p) \end{pmatrix} = \begin{pmatrix} 0 & (n_c - 1)s & n_p\hat{s} \\ s & (n_c - 2)s & n_p\hat{s} \\ \hat{s} & (n_c - 1)\hat{s} & 0 \end{pmatrix} \begin{pmatrix} k^{t-1}(i_c, i_c) \\ k^{t-1}(i_c, j_c) \\ k^{t-1}(i_c, j_p) \end{pmatrix}
$$

and $k^t(i_p, i_p) = k^t(i_p, j_p) = n_c\hat{s}k^t(i_c, j_p)$, where $k^0(i_c, j_c) = 0$, $k^0(i_c, j_p) = 0$, $k^0(i_c, i_c) = 1$. This is a homogenous system of difference equation with initial conditions $k^1(i_c, j_c) = s$, $k^1(i_c, j_p) = \hat{s}$, $k^1(i_c, i_c) = 0$. So, to solve it suffices to derive the eigenvalues of the matrix of coefficients and the respective eigenvectors. To derive eigenvalues note that

$$
\begin{vmatrix} -\lambda & (n_c - 1)s & n_p\hat{s} \\ s & (n_c - 2)s - \lambda & n_p\hat{s} \\ \hat{s} & (n_c - 1)\hat{s} & -\lambda \end{vmatrix} = \begin{vmatrix} -\lambda - s & 0 & 0 \\ s & (n_c - 1)s - \lambda & n_p\hat{s} \\ \hat{s} & n_c\hat{s} & -\lambda \end{vmatrix} = (-\lambda - s) \begin{vmatrix} (n_c - 1)s - \lambda & n_p\hat{s} \\ n_c\hat{s} & -\lambda \end{vmatrix} = 0
$$

if and only if $\lambda = -s$ or $\lambda^2 - \lambda(n_c - 1)s - n_p n_c \hat{s}^2 = 0$. Call $\lambda_1 = -s$, and $\lambda_2 > \lambda_3$ the two solutions to the quadratic equation. Let the eigenvector associated to λ_i be denoted by $v_i = [x_i, y_i, z_i]$. Simple calculation implies that $v_1 = [x_1, -x_1/(n_c - 1), 0]$ and $v_2 = [x_2, x_2, n_c\hat{s}x_2/\lambda_2]$, $v_3 = [x_3, x_3, n_c\hat{s}x_3/\lambda_3]$. Recalling that

$$
\begin{aligned}
k^t(i_c, i_c) &= c_1 x_1 \lambda_1^t + c_2 x_2 \lambda_2^t + c_3 x_3 \lambda_3^t, \\
k^t(i_c, j_c) &= c_1 y_1 \lambda_1^t + c_2 y_2 \lambda_2^t + c_3 y_3 \lambda_3^t, \\
k^t(i_c, j_p) &= c_1 z_1 \lambda_1^t + c_2 z_2 \lambda_2^t + c_3 z_3 \lambda_3^t,
\end{aligned}
$$

and using the derived eigenvalues and eigenvectors we obtain

$$
\begin{aligned}
k^t(i_c, i_c) &= c_1 x_1 (-s)^t + c_2 x_2 \lambda_2^t + c_3 x_3 \lambda_3^t, \\
k^t(i_c, j_c) &= -c_1 x_1 \frac{1}{n_c - 1}(-s)^t + c_2 x_2 \lambda_2^t + c_3 x_3 \lambda_3^t, \\
k^t(i_c, j_p) &= c_2 n_c \hat{s} x_2 \lambda_2^{t-1} + c_3 n_c \hat{s} x_3 \lambda_3^{t-1}.
\end{aligned}
$$

Imposing the initial conditions, we obtain $c_1 x_1 = (n_c - 1)/n_c$, $c_2 x_2 = \frac{1}{n_c}\left[\frac{(n_c-1)s-\lambda_3}{\lambda_2-\lambda_3}\right]$ and $c_3 x_3 = \frac{1}{n_c}\left[\frac{\lambda_2-(n_c-1)s}{\lambda_2-\lambda_3}\right]$. And so, after some algebra,

$$
\begin{aligned}
k^t(i_c, i_c) &= \frac{n_c - 1}{n_c}(-1)^t s^t + \frac{1}{(\lambda_2 - \lambda_3)n_c}\left[(n_c - 1)s(\lambda_2^t - \lambda_3^t) - \lambda_2\lambda_3(\lambda_2^{t-1} - \lambda_3^{t-1})\right], \\
k^t(i_c, j_c) &= -\frac{1}{n_c}(-1)^t s^t + \frac{1}{(\lambda_2 - \lambda_3)n_c}\left[(n_c - 1)s(\lambda_2^t - \lambda_3^t) - \lambda_2\lambda_3(\lambda_2^{t-1} - \lambda_3^{t-1})\right], \\
k^t(i_c, j_p) &= \frac{n_c \hat{s}}{(\lambda_2 - \lambda_3)n_c}\left[(n_c - 1)s(\lambda_2^{t-1} - \lambda_3^{t-1}) - \lambda_2\lambda_3(\lambda_2^{t-2} - \lambda_3^{t-2})\right].
\end{aligned}
$$

We can now derive the matrix Γ. Note that

$$
\sum_{t=1}^{\infty} k^t(i_c, i_c) = \frac{(n_c - 1)s^2 + n_c n_p \hat{s}^2 s + n_p \hat{s}^2}{(s + 1)[1 - s(n_c - 1) - n_c n_p \hat{s}^2]},
$$

and since $\gamma_{i_c, i_c} = (1 - d_c)[1 + \sum_{t=1}^{\infty} k^t(i_c, i_c)]$ we have

$$
\gamma_{i_c, i_c} = \frac{[1 - (n_c - 1)s - n_p \hat{s}][1 - (n_c - 2)s - n_c n_p \hat{s}^2 + n_p \hat{s}^2]}{(s + 1)[1 - s(n_c - 1) - n_c n_p \hat{s}^2]}.
$$

We can repeat the same steps for the other cases and straight algebra leads to following expressions that we have reports in example 1. Furthermore, if we set $n_p = 0$ we get the Γ for the complete network with $n_c = n$ nodes. If we set $n_p = n - 1$, we get the Γ for the star network.

The details of the derivations in examples 2 and 3 are now provided.

Example 2: Complete Network. The formal statement of the result for the complete network is: *Assume that $\mu_i = \mu$, $\sigma_i^2 = \sigma^2$ and $w_i = w$, for all $i \in \mathcal{N}$. In a complete network, an increase in s increases the investment in risky asset of each individual, the expected value of each individual, the variance of each individual and the utility of each individual.*

Proof: Setting $n_p = 0$ and calling $n_c = n$, we get the regular network, with link strength

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$s \le 1/(n-1)$. The element of Γ are therefore $\gamma_{ij} = s/(s+1)$ and $\gamma_{ii} = 1-(n-1)\gamma_{ij}$. Individual investment is negatively related to γ_{ii}, which is clearly decreasing in s, for $s \in [0, 1/(n-1)]$. Next, observe that

$$E[V_i] = wr \sum_{j \in \mathcal{N}} \gamma_{ij} + \frac{(\mu - r)^2}{\alpha \sigma^2} \sum_{j \in \mathcal{N}} \frac{\gamma_{ij}}{\gamma_{jj}} = wr + \frac{(\mu - r)^2}{\alpha \sigma^2} \left[\frac{1+s}{1-(n-2)s} \right].$$

It is straightforward to see that $E[V_i]$ is increasing in s. Similar computation shows that

$$Var[V_i] = \frac{(\mu - r)^2}{\alpha^2 \sigma^2} \left[1 + \frac{(n-1)s^2}{[1-(n-2)s]^2} \right],$$

and it is immediate to see that it is increasing in s. Next, the expected utility of i reads as

$$U_i = E[V_i] - \frac{\alpha}{2} Var[V_i] = wr + \frac{(\mu - r)^2}{2\alpha \sigma^2} \left[1 + \frac{s[2 - s(2n-3)]}{[1-(n-2)s]^2}(n-1) \right],$$

and

$$\frac{\partial U_i}{\partial s} = \frac{(n-1)(\mu - r)^2}{\alpha \sigma^2} \frac{[1 - s(n-1)]}{[1 - (n-2)s]^3} > 0,$$

where the last inequality follows by noticing that, by assumption, $s(n-1) < 1$. Finally, it is easy to check the result on the covariance. This completes the proof. ∎

Example 3: Star Network. The formal statement of the result for the star network is: *Assume that $\mu_i = \mu$, $\sigma_i^2 = \sigma^2$ and $w_i = w$, for all $i \in \mathcal{N}$. Consider the star network with strength of link $\hat{s}$.*

1. *The central individual invests in the risky project more than the other individuals. Furthermore, an increase in $\hat{s}$ increases the investment in the risky asset of each individual.*

2. *The expected value and the variance of each peripheral player (central player) is increasing (non-monotonic) in $\hat{s}$.*

3. *The expected utility of the central player and peripheral players is non-monotonic in $\hat{s}$.*

4. *The covariance between the central player and a peripheral player is higher then the covariance between two peripheral players if $\hat{s}$ is low, and the reverse holds when $\hat{s}$ is high.*

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Proof: Part 1 follows by inspection of the net ownership expressions derived for the star network. To prove part 2 we first consider the difference between the variance of the central player and the variance of a peripheral player; we then note that this is a continuous function of $\hat{s}$ and we derive the sign of such difference for the case where $\hat{s}$ converges to 0 and the case where $\hat{s}$ converges to $1/n_p$. We do the same for the difference between the expected of the central player and the expected returns of a peripheral player. We replicate the same logic to prove part 3.

Using the expression for $Var[V_{i_c}]$ and $Var[V_{i_p}]$ we obtain

$$\Delta_{Var} = Var[V_{i_c}] - Var[V_{i_p}] = \frac{(\mu - r)^2}{\alpha^2 \sigma^2} \left[n_p \left(\frac{\gamma_{i_c i_p}}{\gamma_{i_p i_p}} \right)^2 - \left(\frac{\gamma_{i_p i_c}}{\gamma_{i_c i_c}} \right)^2 - (n_p - 1) \left(\frac{\gamma_{i_p j_p}}{\gamma_{j_p j_p}} \right)^2 \right],$$

and using the expressions for Γ we obtain

$$\Delta_{Var} = \frac{(\mu - r)^2}{\alpha^2 \sigma^2} \hat{s}^2 q(n_p, \hat{s})^2 f(n_p, \hat{s}),$$

where $q(n_p, \hat{s}) = 1/[(1 - \hat{s})(1 - (n_p - 1)\hat{s}^2)(1 - n_p \hat{s})]$ and

$$f(n_p, \hat{s}) = n_p (1 - n_p \hat{s})^4 - (n_p - 1)(1 - n_p \hat{s})^2 \hat{s}^2 (1 - \hat{s})^2 - (1 - \hat{s})^4 [1 - (n_p - 1)\hat{s}^2]^2.$$

Note that $\lim_{\hat{s} \to 0} f(n_p, \hat{s}) = n_p - 1 > 0$ and therefore, for the above expression, it follows that $\lim_{\hat{s} \to 0} \Delta_{Var} = 0$ and that $\Delta_{Var} > 0$ for $\hat{s}$ sufficiently close to 0. Moreover, it is immediate to see that $\lim_{\hat{s} \to 1/n_p} \Delta_{Var} = -\infty$. The sign of Δ_{Var} is dictated by the sign of $f(n_p, \hat{s})$. It is easy to see that $f(n_p, \hat{s})$ is positive at $\hat{s} = 0$ and negative at $\hat{s} = 1/n_p$. It is easy to see that it changes sign only once.

We now turn to the expected value. Using the expressions for $E[V_{i_c}]$ and $E[V_{i_p}]$ we obtain

$$
\begin{aligned}
\Delta_E = E[V_{i_c}] - E[V_{i_p}] &= wr[\gamma_{i_c i_c} + n_p \gamma_{i_c i_p} - \gamma_{i_p i_p} - \gamma_{i_p i_c} - (n_p - 1)\gamma_{i_p j_p}] + \\
&+ \frac{(\mu - r)^2}{\alpha \sigma^2} \left[n_p \frac{\gamma_{i_c i_p}}{\gamma_{i_p i_p}} - \frac{\gamma_{i_p i_c}}{\gamma_{i_c i_c}} - (n_p - 1) \frac{\gamma_{i_p j_p}}{\gamma_{j_p j_p}} \right] \\
&= wr(n_p - 1)[\gamma_{i_c i_p} - \gamma_{i_p i_c}] + \frac{(\mu - r)^2}{\alpha \sigma^2} \left[n_p \frac{\gamma_{i_c i_p}}{\gamma_{i_p i_p}} - \frac{\gamma_{i_p i_c}}{\gamma_{i_c i_c}} - (n_p - 1) \frac{\gamma_{i_p j_p}}{\gamma_{j_p j_p}} \right],
\end{aligned}
$$

where the third equality is obtained by using the property that $\sum_j \gamma_{ji} = 1$ for all i. Using the

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expressions for Γ we obtain

$$\Delta_E = \hat{s}\left[-wrg(n_p, \hat{s}) + \frac{(\mu - r)^2}{\alpha\sigma^2}q(n_p, \hat{s})h(n_p, \hat{s})\right],$$

where $g(n_p, \hat{s}) = \hat{s}(n_p - 1)^2/(1 - n_p\hat{s}^2)$ and

$$h(n_p, \hat{s}) = n_p(1 - n_p\hat{s})(1 - n_p\hat{s} - \hat{s} + \hat{s}^2) - (1 - \hat{s})[(1 - \hat{s})(1 - (n_p - 1)\hat{s}^2) - \hat{s}(1 - n_p\hat{s})].$$

Noticing that $\lim_{\hat{s}\to 0} g(n_p, \hat{s}) = 0$ and $\lim_{\hat{s}\to 0} h(n_p, \hat{s})$ is a positive constant, it follows by inspection of Δ_E that $\lim_{\hat{s}\to 0} \Delta_E = 0$ and that Δ_E is positive for $\hat{s}$ sufficiently small. Similarly, noticing that $\lim_{\hat{s}\to 1/n_p} g(n_p, \hat{s})$ is a positive constant and that $\lim_{\hat{s}\to 1/n_p} h(n_p, \hat{s})$ is a negative constant, it follows by inspection of Δ_E that $\lim_{\hat{s}\to 1/n_p} \Delta_E = -\infty$ and that Δ_E is negative for $\hat{s}$ sufficiently close to $1/n_p$.

We, next, consider the difference in individual optimum utilities. We have that

$$
\begin{aligned}
\Delta_U &= U_{i_c} - U_{i_p} = \Delta_E - \frac{\alpha}{2}\Delta_{Var} \\
&= \hat{s}\left[-wrg(n_p, \hat{s}) + \frac{(\mu - r)^2}{\alpha\sigma^2}q(n_p, \hat{s})h(n_p, \hat{s})\right] - \frac{1}{2}\frac{(\mu - r)^2}{\alpha\sigma^2}\hat{s}^2 q(n_p, \hat{s})^2 f(n_p, \hat{s}), \\
&= \hat{s}\left[-wrg(n_p, \hat{s}) + \frac{(\mu - r)^2}{\alpha\sigma^2}q(n_p, \hat{s})h(n_p, \hat{s}) - \frac{1}{2}\frac{(\mu - r)^2}{\alpha\sigma^2}\hat{s}q(n_p, \hat{s})^2 f(n_p, \hat{s})\right], \\
&= \hat{s}\left[-wrg(n_p, \hat{s}) + \frac{(\mu - r)^2}{\alpha\sigma^2}q(n_p, \hat{s})\left(h(n_p, \hat{s}) - \frac{1}{2}\hat{s}q(n_p, \hat{s})f(n_p, \hat{s})\right)\right].
\end{aligned}
$$

Taking the property of the functions g and h for the two limiting case of $\hat{s}$ discussed above, and simple computation shows that Δ_U is positive for sufficiently small $\hat{s}$ and for $\hat{s}$ sufficiently close to $1/n_p$.

We finally prove the stamens of the covariance. Using the formula of the covariance in equilibrium we have that

$$Cov(V_{i_c}, V_{i_p}) = \frac{(\mu - r)^2}{\alpha^2\sigma^2}\left[\frac{\gamma_{i_c i_c}\gamma_{i_p i_c}}{\gamma_{i_c i_c}^2} + \frac{\gamma_{i_c i_p}\gamma_{i_p i_p}}{\gamma_{i_p i_p}^2} + (n_p - 1)\frac{\gamma_{i_c j_p}\gamma_{i_p j_p}}{\gamma_{j_p j_p}^2}\right]$$

$$Cov(V_{i_p}, V_{j_p}) = \frac{(\mu - r)^2}{\alpha^2\sigma^2}\left[2\frac{\gamma_{i_p i_p}\gamma_{j_p i_p}}{\gamma_{i_p i_p}^2} + \frac{\gamma_{i_p i_c}\gamma_{j_p i_c}}{\gamma_{i_c i_c}^2} + (n_p - 2)\frac{\gamma_{i_p j_p}^2}{\gamma_{j_p j_p}^2}\right]$$

Using the expression for the different γs, we obtain the expressions reported in the example.

It is easy to check that a. $\lim_{\hat{s}\to 0}[Cov(V_{i_c}, V_{i_p}) - Cov(V_{i_p}, V_{j_p})] = 0$ and $Cov(V_{i_c}, V_{i_p}) - Cov(V_{i_p}, V_{j_p}) > 0$ $\hat{s} \in [0, \epsilon]$ for some $\epsilon > 0$, and b. $\lim_{\hat{s}\to 1/n_p}[Cov(V_{i_c}, V_{i_p}) - Cov(V_{i_p}, V_{j_p})] = -\infty$ and $Cov(V_{i_c}, V_{i_p}) - Cov(V_{i_p}, V_{j_p}) < 0$ for $\hat{s} \in [\frac{1}{n_p} - \epsilon, \frac{1}{n_p}]$ for some $\epsilon > 0$. This concludes the proof. $\blacksquare$

Proof of Proposition 2. We start with the derivation of the second-order approximation of Γ for thin networks. Define the indicator function $\delta_{ij} = 1$, if $i = j$ and $\delta_{ij} = 0$, otherwise. First, note that:

$$
\begin{aligned}
\gamma_{ij} &= (1 - \sum_{p\neq i} s_{pi})\left(\delta_{ij} + s_{ij} + \sum_{p\neq i,j} s_{ip}s_{pj} + \sum_{p\neq i, q\neq j, p\neq q} s_{ip}s_{pq}s_{qj} + ...\right) \\
&\simeq \delta_{ij} + s_{ij} + \sum_p s_{ip}s_{pj} - \sum_p s_{pi}(\delta_{ij} + s_{ij}),
\end{aligned}
$$

which yields to

$$\gamma_{ii} \simeq 1 - \eta_i^{in} + \sum_p s_{ip}s_{pi} \text{ and } \gamma_{ij} \simeq s_{ij} - s_{ij}\eta_i^{in} + \sum_p s_{ip}s_{pj}.$$

We can then write:

$$
\begin{aligned}
\frac{\gamma_{ij}}{\gamma_{jj}} &\simeq \frac{s_{ij} + \sum_p s_{ip}s_{pj} - s_{ij}\sum_p s_{pj}}{1 - \left(\sum_p s_{pj} - \sum_p s_{jp}s_{pj}\right)} \\
&\simeq \left(s_{ij} + \sum_p s_{ip}s_{pj} - s_{ij}\sum_p s_{pj}\right)\left(1 + \sum_p s_{pj} - \sum_p s_{jp}s_{pj} + \left(\sum_p s_{pj}\right)\left(\sum_h s_{hj}\right)\right) \\
&\simeq s_{ij} + \sum_p s_{ip}s_{pj} - s_{ij}\sum_p s_{pj} + s_{ij}\sum_p s_{pj} = s_{ij} + \sum_p s_{ip}s_{pj},
\end{aligned}
$$

and, similarly,

$$\frac{\gamma_{ij}^2}{\gamma_{jj}^2} \simeq s_{ij}^2.$$

Therefore

$$\frac{\gamma_{ij}}{\gamma_{jj}} - \frac{1}{2}\frac{\gamma_{ij}^2}{\gamma_{jj}^2} \simeq s_{ij} + \sum_p s_{ip}s_{pj} - \frac{1}{2}s_{ij}^2$$

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Using expression (7), we obtain that in thin networks, $\sum_i U_i(S) > \sum_i U_i(S')$ if and only if

$$\sum_i \sum_j \left[s_{ij} + \sum_p s_{ip} s_{pj} - \frac{1}{2} s_{ij}^2 \right] > \sum_i \sum_j \left[s'_{ij} + \sum_p s'_{ip} s'_{pj} - \frac{1}{2} s'^2_{ij}, \right]$$

and using the definition of η_i^{in} and η_i^{out}, this condition can be rewritten as condition (11) in the Proposition. ∎

Proof of Corollary 1: If S is more integrated than S' then $\eta_i^{out}(S) \geq \eta_i^{out}(S')$ and the inequality is strict for some i. This implies that moving from S' to S there is a positive first order effect in aggregate utilities. Therefore, for $\bar{s}$ small enough, aggregate utility is higher in S and than S'. Next, the proof of the second part of the Corollary follows by Proposition 2 after noticing that $\eta_i^{out}(S) = \eta_i^{out}(S')$ for all $i \in \mathcal{N}$ and using the definition of $\sigma_{s_i}^2$. ∎

Proof of Proposition 3: Rewriting the objective function of the planner (12) we obtain that

$$W(S) = r \sum_{i \in \mathcal{N}} w_i + \sum_{i \in \mathcal{N}} \beta_i(\mu_i - r) - \frac{1}{2} \sum_{i \in \mathcal{N}} \beta_i \sigma_i^2 \left[\phi + (\alpha - \phi) A_i \right], \qquad (15)$$

where $A_i \equiv \sum_{j \in \mathcal{N}} \gamma_{ji}^2$. Suppose the optimum is interior. Then, under the assumption that projects are independent, we obtain that for every $i \in \mathcal{N}$, the first order condition is

$$(\mu_i - r) - \sigma_i^2 \beta_i \left[\phi + (\alpha - \phi) A_i \right] = 0. \qquad (16)$$

We obtain that the optimal level of investment of the social planner is, for every i,

$$\beta_i^P = \min \left[w_i, \hat{\beta}_i \frac{\alpha}{\phi + (\alpha - \phi) A_i} \right]. \qquad (17)$$

∎

Proof of Proposition 4. Recall that, given a network S, the cost of decentralization is

$$K(S) = \frac{(\mu - r)^2}{\alpha \sigma^2} \left[\sum_{j \in \mathcal{N}} \left(\frac{1}{\sum_{l \in \mathcal{N}} \gamma_{lj}^2} - \frac{1}{\gamma_{jj}} \right) - \frac{1}{2} \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N}} \gamma_{ij}^2 \left(\frac{1}{(\sum_{l \in \mathcal{N}} \gamma_{lj}^2)^2} - \frac{1}{\gamma_{jj}^2} \right) \right].$$

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First, using the approximation of γ_{ij} we derive

$$\sum_{h=1}^{n} \gamma_{hj}^2 \simeq \sum_{h \neq j}^{n} \left(s_{hj} + \sum_p s_{hp} s_{pj} - s_{hj} \eta_j^{in} \right)^2 + \left(1 - \eta_j^{in} + \sum_p s_{jp} s_{pj} \right)^2$$

$$\simeq 1 - 2\eta_j^{in} + 2\sum_p s_{jp} s_{pj} + \left(\eta_j^{in} \right)^2 + \left(\sum_h s_{hj}^2 \right).$$

Therefore

$$\frac{1}{\sum_{h=1}^{n} \gamma_{hj}^2} \simeq 1 + 2\eta_j^{in} - 2\sum_p s_{jp} s_{pj} - \left(\eta_j^{in} \right)^2 - \left(\sum_h s_{hj}^2 \right) + 4\left(\eta_j^{in} \right)^2$$

$$= 1 + 2\eta_j^{in} - 2\sum_p s_{jp} s_{pj} - \left(\sum_h s_{hj}^2 \right) + 3\left(\eta_j^{in} \right)^2,$$

and

$$\frac{1}{\gamma_{jj}} \simeq 1 + \eta_j^{in} - \sum_p s_{jp} s_{pj} + \left(\eta_j^{in} \right)^2.$$

We obtain:

$$\frac{1}{\sum_{h=1}^{n} \gamma_{hj}^2} - \frac{1}{\gamma_{jj}} \simeq \eta_j^{in} - \sum_p s_{jp} s_{pj} + 2\left(\eta_j^{in} \right)^2 - \left(\sum_h s_{hj}^2 \right).$$

On the other hand

$$\frac{1}{\left(\sum_{p=1}^{n} \gamma_{pj}^2 \right)^2} - \frac{1}{\gamma_{jj}^2} \simeq \left[1 + 2\eta_j^{in} - 2\sum_p s_{jp} s_{pj} - \left(\sum_h s_{hj}^2 \right) + 3\left(\eta_j^{in} \right)^2 \right]^2$$

$$- \left[1 + \eta_j^{in} - \sum_{p \neq j} s_{jp} s_{pj} + \left(\eta_j^{in} \right)^2 \right]^2$$

$$\simeq 2\eta_j^{in} - 2\sum_p s_{jp} s_{pj} + 7\left(\eta_j^{in} \right)^2 - 2\left(\sum_h s_{hj}^2 \right).$$

Then

$$\sum_i \sum_j \gamma_{ij}^2 \left(\frac{1}{\left(\sum_{p=1}^n \gamma_{pj}^2\right)^2} - \frac{1}{\gamma_{jj}^2} \right) \simeq \sum_i \sum_{j \neq i} \left(s_{ij} + \sum_p s_{ip}s_{pj} - s_{ij}\eta_j^{in} \right)^2 \left(\frac{1}{\left(\sum_p \gamma_{pj}^2\right)^2} - \frac{1}{\gamma_{jj}^2} \right),$$

and after some algebra we obtain

$$\sum_i \sum_j \gamma_{ij}^2 \left(\frac{1}{\left(\sum_{p=1}^n \gamma_{pj}^2\right)^2} - \frac{1}{\gamma_{jj}^2} \right) \simeq \sum_i \left[2\eta_i^{in} - 2\sum_p s_{ip}s_{pi} + 3(\eta_i^{in})^2 - 2\left(\sum_h s_{hi}^2\right) \right].$$

Putting together these expressions we get an expression for the cost of decentralization:

$$
\begin{aligned}
\frac{K(S)}{\frac{(\mu-r)^2}{\alpha\sigma^2}} &\simeq \sum_j \left[\eta_j^{in} - \sum_p s_{jp}s_{pj} + 2\left(\eta_j^{in}\right)^2 - \left(\sum_h s_{hj}^2\right) \right] - \\
&\quad - \frac{1}{2}\sum_j \left[2\eta_j^{in} - 2\sum_p s_{jp}s_{pj} + 3\left(\eta_j^{in}\right)^2 - 2\left(\sum_h s_{hj}^2\right) \right] \\
&= \frac{1}{2}\sum_j \left(\eta_j^{in}(S)\right)^2.
\end{aligned}
$$

It is now straightforward to complete the proof of Proposition 4. ∎

Proof of Proposition 5:

First-best design problem. We start by considering the first best design problem. Substituting in expression (15) the centralised solution $\beta^P = \{\beta_1^P, ..., \beta_n^P\}$, we obtain that

$$W(S, \beta^P) = r\sum_{i \in \mathcal{N}} w_i + \frac{1}{2}\sum_{i \in \mathcal{N}} \hat{\beta}_i(\mu_i - r)\frac{\alpha}{\phi + (\alpha - \phi)A_i}$$

Recall that $A_i = \sum_{j \in \mathcal{N}} \gamma_{ji}^2$ and therefore A_i only depends on $\{\gamma_{1i}, ..., \gamma_{ni}\}$. Moreover, if we fix i, the expresson

$$\hat{\beta}_i(\mu_i - r)\frac{\alpha}{\phi + (\alpha - \phi)A_i}$$

is declining in A_i. Next note that if, for some i, $\gamma_{li} > \gamma_{ki}$ for some $l \neq i$ and $k \neq i$, then, we can always find a small enough $\epsilon > 0$ so that, by making the local change $\gamma_{li}' = \gamma_{li} - \epsilon$ and $\gamma_{ki}' = \gamma_{ki} + \epsilon$, we strictly decrease A_i, without altering A_j for all $j \neq i$. Hence, such a local

change strictly increases welfare. This implies that at the optimum $\gamma_{li} = \gamma_{ki}$ for all $l, k \neq i$. Set $\gamma_{li} = \gamma_{ki} = \gamma$; hence, $\gamma_{ii} = 1 - (n-1)\gamma$. Then, W is maximized when A_i is minimized, or, equivalently, γ minimizes

$$(n-1)\gamma^2 + [1 - \gamma(n-1)]^2$$

which implies that $\gamma = 1/n$. Note that Γ such that $\gamma_{ij} = 1/n$ for all i and for all j is obtained when S is complete and $s_{ij} = 1/(n-1)$ for all i and for all $j = \neq i$.

Second-best design problem. Substituting in expression 15 the decentralised solution $\beta^* = \{\beta_1^*, ..., \beta_n^*\}$, we obtain that

$$
\begin{aligned}
W(S, \beta^*) &= r \sum_{i \in \mathcal{N}} w_i + \sum_{i \in \mathcal{N}} \hat{\beta}_i (\mu_i - r) \frac{1}{\gamma_{ii}} \left[1 - \frac{1}{2} \frac{\phi + (\alpha - \phi) A_i}{\alpha \gamma_{ii}} \right] \\
&= r \sum_{i \in \mathcal{N}} w_i + \sum_{i \in \mathcal{N}} \hat{\beta}_i (\mu_i - r) \Psi_i(\gamma_{1i}, ..., \gamma_{ni})
\end{aligned}
$$

where $\Psi_i(\gamma_{1j}, ..., \gamma_{nj}) \equiv \frac{1}{\gamma_{ii}} \left[1 - \frac{1}{2} \frac{\phi + (\alpha - \phi) A_i}{\alpha \gamma_{ii}} \right]$.

Similarly to the first-best design case, if there exists i such that $\gamma_{ki} > \gamma_{li}$ for some $k, l \neq i$, then we can find a small $\epsilon > 0$ so that the local change $\gamma'_{ki} = \gamma_{ki} - \epsilon$ and $\gamma'_{li} = \gamma_{li} + \epsilon$ reduces A_i, without changing anything else. This, then, leads to an increase in $\Psi_i(\gamma_{1j}, ..., \gamma_{nj})$ and therefore an increase in W. So, at the optimum, for all i, $\gamma_{ki} = \gamma_{li}$ for all $k, l \neq i$.

Using $\gamma_{ki} = \gamma_{li}$ for all $k, l \neq i$ and that Γ is column stochastic we can rewrite $A_i = \frac{(1-\gamma_{ii})^2 + \gamma_{ii}^2(n-1)}{n-1}$ and, then rewrite Ψ_i as a function of γ_{ii}:

$$\Psi_i(\gamma_{ii}) \equiv \frac{1}{2\alpha(n-1)\gamma_{ii}^2} \left[2\alpha\gamma_{ii}(n-1) - \phi(n-1) - (\alpha - \phi)[(1-\gamma_{ii})^2 + \gamma_{ii}^2(n-1)] \right],$$

or

$$\Psi_i(\gamma_{ii}) \equiv \frac{1}{\gamma_{ii}} - \frac{\phi}{2\alpha} \frac{1}{\gamma_{ii}^2} - \frac{\alpha - \phi}{2\alpha(n-1)} \frac{(1-\gamma_{ii})^2}{\gamma_{ii}^2} - \frac{\alpha - \phi}{2\alpha}.$$

Taking the derivative with respect to γ_{ii} we have

$$\frac{\partial \Psi_i(\gamma_{ii})}{\partial \gamma_{ii}} = -\frac{1}{\gamma_{ii}^2} + \frac{\phi}{\alpha} \frac{1}{\gamma_{ii}^3} + \frac{\alpha - \phi}{\alpha(n-1)} \frac{1 - \gamma_{ii}}{\gamma_{ii}^3},$$

and solving $\frac{\partial \Psi_i(\gamma_{ii})}{\partial \gamma_{ii}} = 0$ we obtain $\gamma_{ii} = \frac{\alpha + \phi(n-2)}{\alpha n - \phi}$. It is also easy to check that $\Psi_i(\gamma_{ii})$ is concave at $\gamma_{ii} = \frac{\alpha + \phi(n-2)}{\alpha n - \phi}$. Finally, a network Γ where $\gamma_{ii} = \frac{\alpha + \phi(n-2)}{\alpha n - \phi}$ for all i and $\gamma_{ji} = \gamma_{ki}$ for all $j, k \neq i$, can be obtained with a cross-holding network S such that $s_{ij} = \frac{1}{n-1} \frac{\alpha - \phi}{\alpha}$ for all i

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and for all $j \neq i$. ∎

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Ownership and Control

We discuss the case in which ownership leads to control in decision making. For simplicity we assume that $w_i = w$, $\mu_i = \mu$ and $\sigma_i^2 = \sigma^2$ for all $i \in \mathcal{N}$. We assume that γ_{ij} signifies that individual i has control over a percentage γ_{ij} of j's initial endowment w. That is, individual i unilaterally decides the investment of $\gamma_{ij}w$. We propose two natural scenario and study the consequences for optimal risk taking.

First Scenario. Assume that individual i can invest $\gamma_{ij}w$ in the risk-free asset or in risky project i. In this case, $\gamma_{ij}w$ is a transfer from j to i that occurs before shocks' realization. Hence, Γ simply re-defines initial endowment of individuals: if we start from a situation where $w_i = w$ for all i, then Γ leads that a new distribution $\hat{w} = \{\hat{w}_1, ..., \hat{w}_n\}$ of endowment, where $\hat{w}_i = \sum_j \gamma_{ij}w$. Since, under mean-variance preferences, initial endowment doest not affect the portfolio choice of an individual (as long as the solution is interior), the network S plays no major role in the analysis.

Second Scenario. Suppose that individual i can invest $\gamma_{ij}w$ in the risk free asset or in risky project j. This is a model where γ_{ij} conveys control to i over $\gamma_{ij}w$, but the control is local, in the sense that individual i can only invest $\gamma_{ij}w$ in risky project j. In this case, individual i chooses $\beta_i = \{\beta_{i1}, ..., \beta_{in}\}$, where β_{ij} is the investment in risky project j of endowment $w_{ij} = \gamma_{ij}w$. Of course, $\beta_{ij} \in [0, \gamma_{ij}w]$. It is immediate that individual i's optimal investment is

$$\beta_{ij} = \max \in \left\{ \gamma_{ij}w, \frac{(\mu - r)}{\alpha\sigma_j^2} \right\}.$$

For a given Γ, takes w sufficiently high so that β_{ij} is interior for all ij. This is always possible. Then we can calculate sum of utilities

$$U_i = wr \sum_j \gamma_{ij} + \frac{1}{2}\frac{(\mu - r)^2}{\alpha} \sum_j \frac{1}{\sigma_j^2},$$

and therefore

$$\sum_i U_i = n \left[wr + \frac{1}{2}\frac{(\mu - r)^2}{\alpha} \sum_j \frac{1}{\sigma_j^2} \right].$$

We now observe that this outcome is equivalent to the outcome choosen by a planner with

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mean variance utility with regard to aggregate output. Indeed,

$$V_i = \sum_j \gamma_{ij} wr + \sum_j \gamma_{ij} \beta_{ij}(z_j - r),$$

and

$$V = nwr + \sum_i \sum_j \gamma_{ij} \beta_{ij}(z_j - r).$$

the optimal investment plan of the planner that maximizes $E[V] - \frac{\alpha}{2} Var[V]$ is then the same as the decentralized solution derived above.

Correlations

In this Appendix we relax the assumption that the returns of projects are uncorrelated. We show existence of an equilibrium, we characterise interior equilibria, and we provide sufficient conditions for uniqueness and for existence of an interior equilibrium. We then provide an example in the extreme case of positive perfect correlation which shows that in asymmetric networks some individuals over-invest in risk taking with similar consequences illustrated in our benchmark model with uncorrelated projects.

Recall that each project z_i is normally distributed with mean μ_i and variance σ_i^2 and therefore $z = \{z_1, ..., z_n\}$ is a multivariate normal distribution. Let Ω be the covariance matrix. Under the assumption that z is a non-degenerate multivariate normal distribution, it follows that Ω is positive definite. Let $\circ$ be the Hadamard product. Let also $\mathbf{b}$ be a n dimensional vector where the i-th element is $(\mu_i - r)$

Proposition 6 *There always exists an equilibrium and the equilibrium is unique if $\sum_j s_{ij} < 1/2$ for all $i \in \mathcal{N}$. Furthermore, there exists a $\bar{s} > 0$ so that if $||S||_{max} < \bar{s}$ the unique equilibrium is interior and takes the following form $\beta = \{\beta_1, ..., \beta_n\}$:*

$$\beta = \frac{1}{\alpha}[\Gamma \circ \Omega]^{-1}\mathbf{b}.$$

Proof Proposition 6. Recall that

$$U_i(\beta_i, \beta_{-i}) = \sum_{j \in \mathcal{N}} \gamma_{ij}(wr + \beta_j(\mu_i - r)) - \frac{\alpha}{2} \sum_{j \in \mathcal{N}} \sum_{j' \in \mathcal{N}} \gamma_{ij} \beta_j \gamma_{ij'} \beta_{j'} \sigma_{jj'}^2, \tag{18}$$

and note that $U_i(\beta_i, \beta_{-i})$ is continuous in (β_i, β_{-i}) and it is concave in β_i. Moreover, the

strategy space is from a convex and bounded support. Hence, existence follows from Rosen 1965.

The sufficient condition for uniqueness also follows from Rosen 1965. For some positive vector r, let $g(\beta, r)$ be a vector where element i is $r_i \frac{\partial U_i}{\partial \beta_i}$. Let $G(\beta, r)$ be the Jacobian of $g(\beta, r)$. Rosen (1965) shows that a sufficient condition for uniqueness is that there exists a positive vector r such that for every β and β' the following holds

$$(\beta - \beta')^T g(\beta', r) + (\beta' - \beta)^T g(\beta, r) > 0.$$

Moreover, a sufficient condition for the above condition to hold is that there exists a positive vector r such that the symmetric matrix $G(\beta, r) + G(\beta, r)^T$ is negative definite. In our case, by fixing r to be the unit vector, we have that

$$G(\beta, 1) + G(\beta, 1)^T = -\alpha[\Gamma + \Gamma^T] \circ \Omega$$

So, it is sufficient to show that $[\Gamma + \Gamma^T] \circ \Omega$ is positive definite. It is well known that the Hadamard product of two positive definite matrix is also a positive definite matrix. Since Ω is positive definite, it is sufficient to show that $[\Gamma + \Gamma^T]$ is positive definite. Since the sum of positive definite matrix is a positive definite matrix, it is sufficient to show that Γ is positive definite. The condition that $\sum_j s_{ij} < 1/2$, implies that Γ is a strictly diagonally dominant, and therefore positive definite.

The characterization of an interior equilibrium follows by taking the FOCs. It remains to show that there exists a $\bar{s} > 0$ so that if $||S||_{max} < \bar{s}$ the equilibrium is interior. For this note that taking the derivative of $U_i(\beta_i, \beta_{-i})$ with respect to β_i we have

$$\frac{\partial U_i(\beta_i, \beta_{-i})}{\partial \beta_i} = \gamma_{ii} \left[(\mu_i - r) - \alpha \sum_j \gamma_{ij} \beta_j \sigma_{ji}^2 \right]. \tag{19}$$

If the equilibrium is non-interior, then there exists a i with $\beta_i = 0$ which implies that

$$(\mu_i - r) - \alpha \sum_{j \neq i} \gamma_{ij} \beta_j \sigma_{ji}^2 \leq 0,$$

but as we take $||S||_{max}$ smaller and smaller we have that $\sum_{j \neq i} \gamma_{ij} \beta_j \sigma_{ji}^2$ becomes as small as we wish and therefore we get a contradiction (we can do that because we can make each γ_{ij} small enough for each $i \neq j$ and because β_j is bounded above by w).

■

Perfectly positively correlated projects

Suppose now that projects are positively perfectly correlated. This environment is equivalent to assume that there is only one risky project and all individuals can invest in such project.[12] Recall that $\eta_i^{in} = \sum_j s_{ji}$

Proposition 7 *Assume projects are perfectly positive correlated and that $\mu_i = \mu$ for all $i \in \mathcal{N}$. An interior equilibrium exists if, and only if, $\frac{1}{1-\eta_i^{in}} - \sum_j \frac{s_{ij}}{1-\eta_j^{in}} > 0$ for all i. In an interior equilibrium the following holds*

$$\beta_i = \frac{\mu - r}{\sigma^2 \alpha} \left[\frac{1}{1 - \eta_i^{in}} - \sum_j \frac{s_{ij}}{1 - \eta_j^{in}} \right], \tag{20}$$

$$E[U_i] = wr \sum_j \gamma_{ij} + \frac{1}{2} \frac{(\mu - r)^2}{\alpha \sigma^2}$$

$$\sum E[U_i] = n \left[wr + \frac{1}{2} \frac{(\mu - r)^2}{\alpha \sigma^2} \right]$$

$$\sum_i \beta_i = n \frac{\mu - r}{\alpha \sigma^2}$$

Proof of Proposition 7 The characterization of equilibrium behavior follows immediately from the general analysis by setting $\sigma_{ij}^2 = \sigma^2$ for all i, j. It is immediate to check that in an interior equilibrium $\sum_i \beta_i = n \frac{\mu - r}{\alpha \sigma^2}$. Furthermore, the condition for interior equilibrium follows from inspection of expression (20). We next derive the expression for the equilibrium expected utility of player i. Recall that $E[V_i] = wr \sum_j \gamma_{ij} + (\mu - r) \sum_j \gamma_{ij} \beta_j$; in an interior equilibrium we have that, for every i, $\mu - r - \alpha \sigma^2 \sum_j \gamma_{ij} \beta_j = 0$, and, therefore,

$$E[V_i] = wr \sum_j \gamma_{ij} + \frac{(\mu - r)^2}{\alpha \sigma^2}.$$

---

[12]In fact, the insights we provide in this section will also carry over to an environment where there are n assets, whose returns are i.i.d, and each individual can invest in each of these assets. In the equilibrium of these models, each individual i will choose a total investment in risky assets, say β_i, and then spread such investment equally across the n assets, i.e., individual i invests $\beta_{is} = \beta_i / n$ on each asset s. It is easy to show that the equilibrium investment β_i in this model with n assets is the same as the one that we derived here for one asset.

Similarly, $Var[V_i] = \sigma^2 \left[\sum_j \gamma_{ij}\beta_j\right]^2$, and using the equilibrium conditions we have that:

$$Var[V_i] = \frac{(\mu - r)^2}{\alpha^2\sigma^2}.$$

The equilibrium expected utility of i is therefore

$$
\begin{aligned}
E[U_i] &= wr\sum_j \gamma_{ij} + \frac{(\mu - r)^2}{\alpha\sigma^2} - \frac{\alpha}{2}\frac{(\mu - r)^2}{\alpha^2\sigma^2} \\
&= wr\sum_j \gamma_{ij} + \frac{1}{2}\frac{(\mu - r)^2}{\alpha\sigma^2}.
\end{aligned}
$$

This concludes the proof of Proposition 7. ∎

Note that in an interior equilibrium each individual is exposed to the same amount of risky investment in the sense that for each individual i and j it must be the case that $\sum_l \gamma_{il}\beta_l = \sum_l \gamma_{jl}\beta_l$. Furthermore, this amount is the same as the one that an individual will choose in isolation, i.e., $\sum_l \gamma_{il}\beta_l = \sum_l \gamma_{jl}\beta_l = \frac{(\mu-r)^2}{\alpha\sigma^2}$. This fact, together with the fact that Γ is column stochastic, implies that the sum of risky investment across individuals equals the sum of investment in the risky asset across individuals in the case where the network is empty.

We can easily extend the proposition to non-interior equilibrium. For a given β let $N(\beta)$ be the set of individuals strictly investing in the risky asset, while $\hat{N}(\beta) = N\backslash N(\beta)$. Define $\Gamma_{-\hat{N}(\beta)}$ be the matrix where we delete the rows and the columns corresponding to the individuals in $\hat{N}(\beta)$. So, $\Gamma_{-\hat{N}(\beta)}$ is a $|N(\beta)| \times |N(\beta)|$ matrix. Furthermore, let x be a vector of dimension $|N(\beta)|$.

REMARK: Non-interior equilibria. The profile β is an equilibrium if and only if: a. for all $i \in N(\beta)$, it holds that $\beta_i = x_i$, where $x_i = \frac{\mu-r}{\alpha\sigma^2}[\Gamma_{-\hat{N}(\beta)}]^{-1}\mathbf{1}$ and b. for all $i \in \hat{N}(\beta)$, it holds that $\sum_j \gamma_{ij}\beta_j > \frac{\mu-r}{\alpha\sigma^2}$. Note that in an equilibrium β, we have that $\sum_j \gamma_{ij}\beta_j = \frac{(\mu-r)^2}{\alpha\sigma^2}$ for all $i \in N(\beta)$; furthermore, it also hods that $\sum_i \beta_i \geq n\frac{\mu-r}{\alpha\sigma^2}$, where the equality holds if and only if the equilibrium is interior.

From the explicit characterization of Proposition 7, it is easy to provide the following comparative statics:

Proposition 8 *Assume projects are perfectly positive correlated. In an interior equilibrium a change in the network has not impact on aggregate utilities and a change in the network increases the utility of individual i if and only if it increases his total ownership $\sum_j \gamma_{ij}$. Hence, an increase in integration leads to a Pareto improvement.*

We now turn to social welfare. The first proposition shows that whenever there is an interior equilibrium, the equilibrium investment is also socially efficient.

Proposition 9 *Assume projects are perfectly positive correlated and suppose the equilibrium is interior. Then, equilibrium investments are socially efficient.*

Proof of proposition 9. We first assume that the socially optimal is interior and show that it coincides with the equilibrium behavior. We then show that the social welfare is concave in β. Under perfect positive correlation, we can write the social welfare as

$$W(\beta_i, \beta_{-i}) = nwr + \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N}} \gamma_{ij}\beta_j(\mu - r) - \frac{\alpha\sigma^2}{2} \sum_{i \in \mathcal{N}} \left[\sum_{j \in \mathcal{N}} \gamma_{ij}\beta_j \right]^2.$$

Taking the derivative with respect to β_l we have

$$
\begin{aligned}
\frac{dW}{d\beta_l} &= \sum_{i \in \mathcal{N}} \left[(\mu - r)\gamma_{il} - \alpha\sigma^2 \gamma_{il} \sum_{j \in \mathcal{N}} \gamma_{ij}\beta_j \right] \\
&= \mu - r - \alpha\sigma^2 \sum_{i \in \mathcal{N}} \gamma_{il} \sum_{j \in \mathcal{N}} \gamma_{ij}\beta_j,
\end{aligned}
$$

and given the assumption that the optimum is interior, we have that for every l it must to hold

$$\sum_{i \in \mathcal{N}} \gamma_{il} \sum_{j \in \mathcal{N}} \gamma_{ij}\beta_j = \frac{\mu - r}{\alpha\sigma^2}.$$

Note that the equilibrium solution derived in Proposition 7 also solves the above problem. Indeed, the equilibrium solution has the property that, for every i, $\sum_j \gamma_{ij}\beta_j = \frac{\mu-r}{\alpha\sigma^2}$. It is also immediate to see that there is a unique solution to the above linear system.

We now show that the social welfare is concave in β. To see this note that the Hessian is $-\sigma^2\alpha\Gamma^T\Gamma$, and therefore we need to show that $\Gamma^T\Gamma$ is positive definite. This is true because

$$x^T\Gamma^T = \{x_1\gamma_{11}, .., x_n\gamma_{1n}; ....; x_1\gamma_{n1}, ..., x_n\gamma_{nn}\} \text{ and } \Gamma x = (x^T\Gamma^T)^T$$

and so

$$x^T \Gamma^T \Gamma x = [x_1 \gamma_{11} + .. + x_n \gamma_{1n}]^2 + ... + [x_1 \gamma_{n1}, ..., x_n \gamma_{nn}]^2 > 0,$$

where the strict inequality follows because $x_i \neq 0$ for some i and because for each i, $\gamma_{ij} > 0$ for some j. ∎

Example 5 *Two-individuals case and over-investment.*

We now develop fully the equilibrium investment and the socially optimal investment in the case of two individuals. Note first that, with two agents, S is characterised by s_{12} and s_{21}, and it is immediate to derive: $\gamma_{ii} = (1 - s_{ji})/(1 - s_{ij}s_{ji})$ and $\gamma_{ji} = 1 - \gamma_{ii}$, for all $i = 1, 2$ and $j \neq i$. Second, we provide a full characterisation of equilibrium. From Proposition 7 we know that an interior equilibrium exists if, and only if, $\frac{1}{1-\eta_i^{in}} - \sum_j \frac{s_{ij}}{1-\eta_j^{in}} > 0$ for all i, which, in this example, reads as $\gamma_{ii} > 1/2$ for $i = 1, 2$. We now characterise non-interior equilibrium. It is immediate to see that the marginal returns to agent i are strictly positive at $\beta_i = \beta_j = 0$; therefore $\beta_1 = \beta_2 = 0$ cannot be equilibrium. Consider the case where $\beta_i = 0$ and $\beta_j > 0$. Given $\beta_i = 0$, the FOC for j leads to

$$\beta_j = \frac{\mu - r}{\alpha \sigma^2} \frac{1 - s_{ij}s_{ji}}{1 - s_{ij}},$$

and since $\beta_i = 0$, it has to be the case that the marginal utility of i at $\beta_i = 0$ and $\beta_j = \frac{\mu-r}{\alpha\sigma^2} \frac{1-s_{ij}s_{ji}}{1-s_{ij}}$ is non-positive, which holds, if, and only if, $\mu - r - \alpha \gamma_{ii} \beta_j \sigma^2 \leq 0$, if, and only if, $\gamma_j j < 1/2$. Combining these results we have that the equilibrium is unique and that there are three regions, which are the same as the one depicted in figure 6. In Region 1, $\gamma_1 1 > 1/2$ and $\gamma_{22} > 1/2$, both agents invest positively in the risky asset and their investment is specified in Proposition 7. In Region 2 (resp. Region 3), where $\gamma_{11} < 1/2$ and $\gamma_{22} > 1/2$ (resp. $\gamma_{11} > 1/2$ and $\gamma_{22} < 1/2$), agent 2 (resp. agent 1)does not invest in the risky project and agent 1 (resp. agent 2) invests $\beta_1 = \frac{\mu-r}{\alpha\sigma^2} \frac{1-s_{21}s_{12}}{1-s_{21}}$ (resp. $\beta_2 = \frac{\mu-r}{\alpha\sigma^2} \frac{1-s_{21}s_{12}}{1-s_{12}}$).

Third, we provide a full characterisation of the social optimum. If the optimum is interior, then we know from Proposition 9 that $\hat{\beta}_i$ equals expression (20), which, in this example reads

$$\hat{\beta}_i = \frac{\mu - r}{\sigma^2 \alpha} \left[\frac{1 - 2s_{ij} + s_{ij}s_{ji}}{(1 - s_{ij})(1 - s_{ji})} \right] \tag{21}$$

and, it is easy to verify that, $\hat{\beta}_1 > 0$ and $\hat{\beta}_2 > 0$ if and only if: $\gamma_{11} > 1/2$ and $\gamma_{22} > 1/2$. The social welfare that is generated is $2wr + \frac{(\mu-r)^2}{\sigma^2 \alpha}$. Consider now that $\hat{\beta}_i = 0$ and $\hat{\beta}_j > 0$. Then
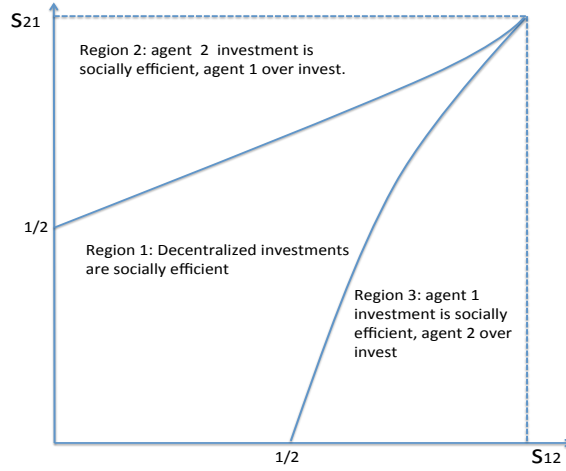
Figure 6: Equilibrium and first best investments.

the FOC for j must hold which leads to

$$\hat{\beta}_j = \frac{\mu - r}{\sigma^2 \alpha} \frac{1}{\gamma_{ij}^2 + \gamma_{jj}^2}$$

and the social welfare generated is $2wr + \frac{(\mu - r)^2}{\sigma^2 \alpha} \frac{1}{2[\gamma_{ij}^2 + \gamma_{jj}^2]}$.

It is easy to see that $2[\gamma_{ij}^2 + \gamma_{jj}^2] > 1$ which implies that the social welfare when $\hat{\beta}_1$ and $\hat{\beta}_2$ are both positive is higher than when one of them is 0. So, whenever $\gamma_{22} > 1/2$ and $\gamma_{11} > 1/2$ the optimal solution is interior. Next, note that the welfare associate to $\hat{\beta}_1 = 0$ and $\hat{\beta}_2 > 0$ is higher than the welfare associated to $\hat{\beta}_1 > 0$ and $\hat{\beta}_2 = 0$ if and only if $\gamma_{12}^2 + \gamma_{22}^2 < \gamma_{21}^2 + \gamma_{11}^2$, which is satisfied if, and only if, $\gamma_{22} < 1/2$ and $\gamma_{11} > 1/2$. Finally, by comparing the optimal investment β_j when $\gamma_{jj} > 1/2$ and $\gamma_{ii} < 1/2$, with the equilibrium investment, it is easy to check that individual j over invest relative to the social planner.

Finally, the comparison between equilibrium and social optimum is summarised in Figure 6. It shows that when the cross-holding network is asymmetric then we have over-investment of some of the agents, which is similar to the case of independent projects.

Endogenous Networks

In this appendix we provide a simple model of network formation. We posit that, simultaneously, each individual i chooses how much to invest in the risky project, β_i, in the risk-free asset, x_i, and the shares/fractions $\gamma_i = \{\gamma_{i1}, ..., \gamma_{in}\}$ to buy of firms fundamental assets $\{w_1, ..., w_n\}$. All this determines: a profile of risky investment β, a profile of risk-free assets investment x and a cross-ownership network Γ.

Given a price vector $p = \{p_1, ..., p_n\}$, where p_i specifies the per-unit price of i's share, we must impose that the choice of each individual is feasible. That is, for each individual i the following constraints must hold:

$$w_i + p_i(1 - \gamma_{ii}) = \beta_i + x_i + \sum_{j \in \mathcal{N} \setminus \{i\}} \gamma_{ij} p_j \tag{22}$$

The LHS is the initial endowment plus the capital that i has obtained by selling his own shares. The RHS is the total investment of individual i.

Given β, x, Γ, and the realisation of each project, we have that $V_i = \sum_j \gamma_{ij} [\beta_j z_j + x_j r]$ and since i has mean-variance preferences over V_i, we have that

$$U_i(\beta, x, \Gamma) = \sum_{j \in \mathcal{N}} \gamma_{ij} [\beta_j \mu_j + x_j r] - \frac{\alpha}{2} \sum_{j \in \mathcal{N}} \gamma_{ij}^2 \beta_j^2 \sigma_j^2 \tag{23}$$

An equilibrium is $(\beta^*, x^*, \Gamma^*, p^*)$ such that, for each i, (1) $(\beta_i^*, x_i^*, \gamma_{i1}^*, ..., \gamma_{in}^*)$ maximises (23) subject to 22 and given $\beta_{-i}^*, x_{-i}^*, p^*$ and (2) the market clears, i.e., $\sum_j \gamma_{ji}^* = 1$ for all i.

Proposition 10 *There exists a $\bar{w}$ so that if $w_i > \bar{w}$ for all $i \in \mathcal{N}$, then there exists an equilibrium that replicates the first best outcome of an utilitarian planner, i.e., $\gamma_{ij}^* = 1/n$ for all $i, j \in \mathcal{N}$, and for all i, $\beta_i^* = \hat{\beta}_i/\gamma_{ii}$ and $p_i^* = \frac{n}{n-1} \sum_{j \neq i} w_j$.*

Proof of Proposition 10. individual i chooses $\gamma_i = \{\gamma_{i1}, ..., \gamma_{in}\}$ and β_i to maximise

$$U_i = \sum_{j \in \mathcal{N}} \gamma_{ij} [\beta_j \mu_j + x_j r] - \frac{\alpha}{2} \sum_{j \in \mathcal{N}} \gamma_{ij}^2 \beta_j^2 \sigma_j^2,$$

where $x_i = w_i + p_i - \beta_i - \sum_{j \in \mathcal{N}} \gamma_{ij} p_j$. Since $x_i \geq 0$, we have that $\beta_i \leq w_i + p_i - \sum_{j \in \mathcal{N}} \gamma_{ij} p_j$.

We now assume that there exists an equilibrium where $\beta_i \leq w_i + p_i - \sum_{j \in \mathcal{N}} \gamma_{ij} p_j$ for all i. Under this assumption, we derive the condition for equilibrium and then we verify that such equilibrium always exists under suitable conditions.

First note that

$$\frac{\partial U_i}{\partial \beta_i} = 0 \text{ iff } \beta_i = \hat{\beta}_i/\gamma_{ii},$$

and

$$\frac{\partial U_i}{\partial \gamma_{ii}} = \beta_i \mu_i + x_i r + \gamma_{ii} r \frac{\partial x_i}{\partial \gamma_{ii}} - \alpha \sigma_i^2 \gamma_{ii} \beta_i^2 \tag{24}$$

$$= r[w_i + p_i(1 - \gamma_{ii}) - \sum_{j \in \mathcal{N}} \gamma_{ij} p_j] = 0, \tag{25}$$

where the second equality follows by using the expression for x_i and using the fact that $\beta_i = \hat{\beta}_i/\gamma_{ii}$, and the third equality is the FOC. Note that the marginal returns of i to self-ownership depends on her initial endowment. Similarly, for every $j \neq i$, we have

$$\frac{\partial U_i}{\partial \gamma_{ij}} = \beta_j \mu_j + x_j r + \gamma_{ii} r \frac{\partial x_i}{\partial \gamma_{ij}} - \alpha \sigma_j^2 \gamma_{ij} \beta_j^2 \tag{26}$$

$$= \beta_j(\mu_j - r)\left(1 - \frac{\gamma_{ij}}{\gamma_{jj}}\right) + r[w_j + p_j - \sum_{j' \in \mathcal{N}} \gamma_{jj'} p_{j'}] - \gamma_{ii} r p_j \tag{27}$$

$$= \beta_j(\mu_j - r)\left(1 - \frac{\gamma_{ij}}{\gamma_{jj}}\right) + r p_j[\gamma_{jj} - \gamma_{ii}] = 0 \tag{28}$$

where the second equality follows because, by rational expectation, $\beta_j = \hat{\beta}_j/\gamma_{jj}$ and $x_j = w_j + p_j - \beta_j - \sum_{j' \in \mathcal{N}} \gamma_{j'} p_{j'}$ (substitute these two expressions in the first line, and then do a bit of simple algebra) and because $\frac{\partial x_i}{\partial \gamma_{ij}} = -p_j$, the third equality follows because for j must also hold the condition that $\partial U_j/\partial \gamma_{jj} = 0$ which is given above.

Hence, an equilibrium is characterised by: $\beta_i = \hat{\beta}_i/\gamma_{ii}$ for all i, and for all i it must also hold that

$$w_i + p_i(1 - \gamma_{ii}) - \sum_{j \in \mathcal{N}} \gamma_{ij} p_j = 0, \tag{29}$$

and for all i and for all $j \neq i$, it must hold that

$$\beta_j(\mu_j - r)\left(1 - \frac{\gamma_{ij}}{\gamma_{jj}}\right) + r p_j[\gamma_{jj} - \gamma_{ii}] = 0. \tag{30}$$

We now claim that for there exists $\bar{w}$ so that if $w_i \geq \bar{w}$ for all i, then there exists an equilibrium where the network is fully symmetric, i.e, $\gamma_{ij} = 1/n$ for all i and j, and we characterise the equilibrium price below. Note that if $\gamma_{ij} = 1/n$ for all i and j, then condition

30 is satisfied. Furthermore, condition 29 becomes

$$w_i + \frac{n-1}{n} p_i - \frac{1}{n} \sum_j p_j = 0. \tag{31}$$

Since this must hold for all $i \in \mathcal{N}$, if we sum up the LHS across i we must get that the sum equals 0, that is

$$\sum_{i \in \mathcal{N}} w_i + \frac{n-1}{n} \sum_{i \in \mathcal{N}} p_i - \sum_{j \in \mathcal{N}} p_j = 0,$$

which implies that $\sum_{j \in \mathcal{N}} p_j = n \sum_j w_j$. Using this condition on 31 and solving for p_i we obtain

$$p_i = \frac{n}{n-1} \left[\sum_j w_j - w_i \right]. \tag{32}$$

We now need to verify that, indeed, $\beta_i \le w_i + p_i - \sum_{j \in \mathcal{N}} \gamma_{ij} p_j$ for all i. Using the expression for p_i and for γ_{ij} the condition becomes

$$\beta_i \;\le\; w_i + p_i - \frac{1}{n} \sum_j p_j = w_i + p_i - \sum_j w_j \tag{33}$$

$$= \; w_i + \frac{n}{n-1} \left[\sum_j w_j - w_i \right] - \sum_j w_j \tag{34}$$

$$= \; \frac{1}{n-1} \left[(n-1)w_i + n \sum_j w_j - n w_i - (n-1) \sum_j w_j \right] \tag{35}$$

$$= \; \frac{1}{n-1} \left[\sum_j w_j - w_i \right] = \frac{1}{n-1} \left[\sum_{j \ne i} w_j \right] \tag{36}$$

$$\tag{37}$$

note that if $w_l \ge \bar{w}$ for all l then

$$\frac{1}{n-1} \left[\sum_{j \ne i} w_j \right] \ge \bar{w}$$

and so it is sufficient that $\beta_i \le \bar{w}$. Since β_i is independent of $\{w_1, ..., w_n\}$, the inequality is satisfied for $\bar{w}$ sufficient high. This concludes the proof of Proposition 10. ∎