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SOME DYNAMIC AND STEADY-STATE PROPERTIES OF THRESHOLD AUTOREGRESSIONS WITH APPLICATIONS TO STATIONARITY AND LOCAL EXPLOSIVITY

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The purpose of this paper is to investigate the dynamics and steady-state properties of threshold autoregressive models with exogenous states that follow Markovian processes; these processes are widely used in applied economics although their statistical properties have not been explored in detail. We use characteristic functions to carry out the analysis and this allows us to describe limiting distributions for processes not considered in the literature previously. We also calculate analytical expressions for some moments. Furthermore, we see that we can have locally explosive processes that are explosive in one regime whilst being strongly stationary overall. This is explored through simulation analysis where we also show how the distribution changes when the explosive state become more frequent although the overall process remains stationary. In doing so, we are able to relate our analysis to asset prices which exhibit similar distributional properties.

Some Dynamic and Steady-State Properties of Threshold Autoregressions with Applications to Stationarity and Local Explosivity

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Abstract

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Keywords: Threshold Auto-regression, Markov process.

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1. Introduction

The purpose of this paper is to investigate the dynamics and steady-state properties of threshold autoregressive models with exogenous states that follow Markovian processes. These models will fall within the class of regime switching models that have become increasingly popular in applied economics and finance. Initially introduced by Goldfeld and Quandt (1973), regime switching models have been used in Economics and finance for a wide variety of applications including forecasting exchange rates (Engel, 1992), understanding price transmission (Goodwin et al, 2000), to find Bubbles in the art market (Knight et al, 2014) and to provide a metric of market efficiency (Ahmed et al, 2018). Hansen (2011) provides a concise summary of threshold autoregressive processes and their applications.

Hamilton (1989, 1990, 2005) has made seminal contributions to the theory and application of regime-switching models. As outlined above, this article discusses a particular class of regime switching models. The problems we discuss appear to have much in common with Markov switching models and Timmermann (2000) has provided a detailed analysis of moments and autocorrelations, which would include our model as MSIII in his terminology. However, his analysis does not address non-moment distributional properties or the non-existence of moment-generating functions (mgfs).

Indeed, Timmermann states in appendix 1, page 103, that "The expressions for the cases where ε_t follows a t-distribution or a normal distribution are based on the moment-generating distributions for these distributions" and this is confusing as it is known that the t-distribution does not have a moment generating function. We therefore re-examine this model, allowing for the non-existence of moment generating functions and use the characteristic function (which will always exist,) to derive certain properties of the model.

Whilst we could carry out a similar analysis for the other models described in Timmermann (2000), our focus is on threshold auto-regression and the elusive search for explicit steady-state distributions. Prior to this article, Gonzalo et al (1997) have described statistical properties of TAR(1) models; however, their analysis is restricted to a mixture of stationary and unit roots whereas our analysis considers non-stationary roots as well. Another important contribution relevant to our work is Pourahmadi (1988); in theorem 3.1 in his article, Pourahmadi(ibid) discusses the covariance stationarity of a process similar to the threshold autoregressive process we consider in section 4 in that he analyses processes with a unit root and a zero root. However, the results derived in section 2 below and the processes considered in section 4 are applicable to a more general setting and we do not restrict ourselves to covariance stationarity.

Caner and Hansen (2001) and Kapetanios and Shin (2003) also consider similar processes but their objective is to derive the distribution of unit root test statistics in the threshold autoregressive framework rather than the distribution of the underlying process. Our results build upon the results of Knight et al (2011) and Ahmed et al (2018); both articles discuss theoretical moments for threshold autoregressive models with exogenous triggers. While Ahmed et al only consider moments when the exogenous variable is independently and identically distributed, Knight et al also consider a Markovian exogenous variable.

In addition to deriving theoretical moments, we also use simulation analysis to show how the distributions of Threshold Autoregressive models change when the process's two states consist of one stationary state and an explosive state. Our analysis focuses on these models for two reasons. Firstly, we are able to derive a characteristic function for this case, thereby adding to the literature on analytical results for threshold autoregressive models and secondly, this class of models is of interest in the financial literature concerned with explosive roots. We also believe that these models can prove to be useful in the applied macroeconomic literature. The simulation analysis presented in this article will help the reader appreciate how these models can be useful in practice.

In applied Macroeconomics for instance, DSGE models often model shocks as AR(1) processes (see Schmitt-Grohe and Uribe, 2004). The literature came under particular scrutiny after the financial crisis for its inability to simulate and thereby predict conditions and outcomes that were observed during the crisis. In addition to the absence of financial markets, such models are also restricted by their reliance on a stationary AR(1) model as a shock process as these processes can rarely be used to study the kind of macroeconomic shocks that led to the financial crisis. On the other hand, these models will not have analytical or numerical solutions if the process was non-stationary.

We postulate that using a TAR(1) shock process which is stationary but nevertheless can exhibit locally non-stationary behaviour can improve these models. Our work will enable calculation of moments for such shocks (where such moments exist), allowing the user to work with analytical solutions or if the user is deriving a numerical solution, ensuring that such a solution will exist. Indeed, some work has already started relying on Markov-switching DSGE models (Foerster et al 2016). This paper complements the proposed methodology by enabling researchers to control and simulate shocks of specific variances.

There are further applications in the Finance literature where TAR models have become popular. The applications may extend to forecasting oil prices through threshold models or in modelling exchange rate fluctuations. There are many areas where a TAR model and the characteristic functions we derive can provide more depth to the underlying analysis. To use one recent example, Aleem et al 2014, estimate a TAR model of exchange-rate pass through for Mexico. Their analysis is limited to an estimation of the threshold above which the pass through is greater. With the aid of characteristic functions from our article, they will have been able to estimate the volatility of exchange rates in their model, improving both their model and the resulting predictions. Similarly, in our earlier article Ahmed et al (ibid), the empirical application relies on a Markovian exogenous trigger; the results from this article will have allowed us to derive moments of our empirical TAR(1). Theorem 5 in Section 2, offers one example of how the results of this article may contribute to applied and empirical work in finance.

The rest of the article is organized as follows. In Section 2, we present the derivation and formulae for characteristic functions of threshold autoregressive(1) models with exogenous Markovian triggers. We also generalize these formulae for Threshold Vector Autoregressive models. Section 3 outlines the simulation methodology. A separate section is necessitated since obtaining a sample from the steady state distribution of a TAR model with a Markov-switching exogenous trigger is a non-trivial exercise. Section 4 presents and discusses simulation results and Section 5 concludes.

2. Moment Generating Functions of TAR(1) Models with Exogenous Markov-Triggers

In this section, we introduce the Threshold Autoregressive model with a Markov-switching exogenous trigger. After introducing the model, we derive the moment generating functions for this model and present some interesting results. We shift to characteristic functions after theorem 3 where we do not have to assume the existence of all moments so that moment generating functions for such processes need not exist. Characteristic functions on the other hand will always exist. Such processes are often used to model prices (particularly in finance), therefore, we refer to our model as a price process indicated by p_t.

The price process has a switching AR(1) form:

$$p_t = \psi_{t-1} + \phi_{t-1} p_{t-1} + \sigma_{t-1}\eta_t \qquad (1)$$

where ψ_{t-1} is a switching drift, ϕ_{t-1} is a switching coefficient term and σ_{t-1} is a switching variance term for the error process. Let

$$\psi_{t-1} =< \alpha, z_{t-1} > \qquad \phi_{t-1} =< \beta, z_{t-1} > \qquad \sigma_{t-1} =< \sigma, z_{t-1} >$$

where α is a vector containing all drift terms, β is a vector of coefficients and σ is a vector of error variances. In general, all the above vectors are k × 1 but we illustrate them when k = 3 for notational convenience. We assume that z_t is Markovian and follows a multinomial distribution and that η_t has a moment generating function $\psi(u)$ which is assumed to be location-scale. In particular,

$$\alpha = (\alpha_1, \alpha_2, \alpha_3)' \quad \beta = (\beta_1, \beta_2, \beta_3)' \quad \sigma = (\sigma_1, \sigma_2, \sigma_3)' \quad \eta_t \sim iid(0,1)$$

$$z_t \in \{e_1, e_2, e_3\} \quad e_1 = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad e_2 = \begin{pmatrix}0\\1\\0\end{pmatrix} \quad e_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$E[z_t|z_{t-1}] = Pz_{t-1}$, where P represents the Transition Matrix for the Markovian state variable z_t. For econometric purposes we envisage an exogenous continuous random variable Z_t and constants $\vartheta_{0,} \dots . \vartheta_{3,}$ so that $z_t = e_j$ if $\vartheta_{j-1,} \le Z_t < \vartheta_{j,}$ i.e. when the continuous random variable Z_t is between thresholds ϑ_{j-1} and ϑ_j, the Markovian variable z_t is equal to e_j.

Here $P = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}$ is the transition matrix, which describes the probability of switching. Also $i'P = i'$ where i is a vector of ones, and $P\pi = \pi$ where π is the vector of stationary(steady − state)probabilities $\mathrm{i}'\pi = 1$.

$$p_{ji} = P\left(z_{t+1} = e_j \middle| z_t = e_i\right) = P\left(z_1 = e_j \middle| z_0 = e_i\right) \qquad 1 \le i, j \le 3$$

so that the Markov Chain is stationary. Whilst we can estimate P by counting frequencies, we can also hypothesise a Markov process for Z_t and then integrate over the appropriate rectangle of the probability density function of $(Z_t \ , Z_{t+1} \ )$

We now consider $\exp(up_t)$, in order to derive the moment generating function for p_t. Here, $u \in \mathbb{R}$.

The moment generating function of p_t is defined by $\phi_t(u) = E[\exp(up_t)]$. Our aim is to determine a recursion for $\phi_t(u) \in R$

Now $z_t = Pz_{t-1} + \nu_t \in R^3$ (2)

where $E[\nu_t | z_{t-1}] = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0 \in R^3$.

From (1) we have that:

$exp\ (up_t) = exp[u < \alpha, z_{t-1} >]\ exp[u\ < \beta, z_{t-1} > p_{t-1}]\ exp\ (u < \ \sigma, z_{t-1} > \eta_t)$

Using iterated expectation,

$\phi_t(u) = E(E(exp[u < \alpha, z_{t-1} >]\ exp[u\ < \beta, z_{t-1} > p_{t-1}]\ exp\ (u < \ \sigma, z_{t-1} > \eta_t / z_{t-1}))$

For functions $F\ (p_{t-1}, Z_{t-1})$, we note that $E(F(p_{t-1}, Z_{t-1})\ ) = E\left(\sum_{j=1}^{k} F(\ p_{t-1}, e_j\right) \pi_j)$ by the law of total probability.

Thus, $\phi_t(u) = \sum_{j=1}^{k} \pi_j(exp\left[u < \alpha, e_j >\right]\ \phi_{t-1}\left[< \beta, e_j > u\right]\ \psi\left(< \sigma, e_j > u\right)\ \Big)$is a dynamic recursion for the mgf of p_t (3)

<u>Steady-State Distribution under Markovian States.</u>

The above discussion leads to the following result.

Theorem 1: Assuming a steady state for prices, denote $E(ex\,p(\ up_t)\ ) = \phi(u)$ and $E(ex\,p(\ u\eta_t)\ ) = \psi(u)$ as the appropriate mgfs (or characteristic functions with a trivial definitional change).

$$\phi(u) = \sum_{j=1}^{k} exp\ \left(u\alpha_j\right)\ \phi\left(\beta_j u\right)\ \pi_j\ \psi\left(\sigma_j u\right) \qquad (4)$$

is the steady-state relationship.

We can use Theorem 1 to arrive at analytical expressions for different moments of the process, p_t.

Define $\mu_B = \sum_{j=1}^{k}\ \pi_j \beta_j \quad ; \mu_{2B} = \sum_{j=1}^{k}\ \pi_j \beta_j^2 \quad ; \quad \sigma_B^2 = \mu_{2B} - (\mu_B)^2$

$\mu_\alpha = \sum_{j=1}^{k} \pi_j \alpha_j$; $\mu_\sigma = \sum_{j=1}^{k} \pi_j \sigma_j$; $\mu_{\alpha B} = \sum_{j=1}^{k} \pi_j \beta_j \alpha_j$ etc.

If we differentiate (4) once to obtain the first moment of p_t, we get

$$E(p_t) = \frac{\mu_\alpha + \mu_\sigma E(\eta_t)}{1-\mu_B} \ ;$$

And, differentiating a second time for the second moment,

$$E(p_t^2) = \frac{\mu_{2\alpha} + 2\mu_{\alpha B}E(p_t) + 2\mu_{\sigma B}E(\eta_t)E(p_t) + 2\mu_{\alpha\sigma}E(\eta_t) + \mu_{2\sigma}E(\eta_t^2)}{1-\mu_{2B}}$$

Further calculations and simplifications lead to an expression for the variance of p_t;

$$Var(p_t) = \mathrm{E}(p_t^2) - \big(E(p_t)\big)^2$$

$$Var(p_t) = \frac{\mu_{2\alpha} + 2\mu_{\alpha B}E\big(p_t\big) + 2\mu_{\sigma B}E(\eta_t)E\big(p_t\big) + 2\mu_{\alpha\sigma}E(\eta_t) + \mu_{2\sigma}E\big(\eta_t^2\big)}{1-\mu_{2B}} - \left(\frac{\mu_\alpha + \mu_\sigma E(\eta_t)}{1-\mu_B}\right)^2$$

Theorem 1 has many corollaries; we list one below but other results can be regarded as special cases. Here we are concerned with the case where k = 2, $\alpha_j = 0, \beta_j = \beta, \sigma_2 = 0/$

$$\phi(u) = \phi(\beta u)(\pi + (1-\pi)\psi(u)) \qquad (5)$$

Corollary 1.

If $\psi(u)$ is the moment generating function of a negative exponential with parameter λ and $\alpha_j = 0, \sigma_1 = \sigma, \sigma_2 = 0$ and $\pi = \beta$ where $\beta_1 = \beta_2 = \beta$ which is less than 1, then $\phi(u) = \frac{\lambda}{\lambda - u}$ i.e. a negative exponential random variable with parameter λ.

Proof:

To show that equation (5) has a solution for some $\psi(u)$, we consider the negative exponential function, i.e. we assume that the disturbance term is distributed as a negative exponential with parameter λ. The corresponding moment generating function, $\phi(u)$, for this disturbance term is $\frac{\lambda}{\lambda - u}$, if we further assume that $\beta = \pi$ and that $0 \le \beta < 1$. Note that equation (5) corresponds to the situation where the β coefficient does not move across states, but the standard deviation of the disturbance term does, i.e. $\sigma_1 = \sigma$ and $\sigma_2 = 0$. Our result implies that $\sigma = \frac{1}{\lambda}$.

For our distributional assumption regarding the disturbance, the corresponding moment generating function is (taking into consideration the two states):

$$\big(\pi + (1-\pi)\psi(u)\big) = \pi + \frac{(1-\pi)\lambda}{\lambda - u}$$

Substituting this in (5) and using the trial solution $\phi(u) = \frac{\lambda}{\lambda - u}$, we have:

$$\frac{\lambda}{\lambda - u} = \left(\frac{\lambda}{\lambda - \beta u}\right)\left(\pi + \frac{(1-\pi)\lambda}{\lambda - u}\right)$$

If we further assume that $\beta = \pi$, the LHS and RHS are equal, thereby proving our result.

We recognise the solution as being a negative exponential auto regression of degree 1 (NEAR(1)); these were investigated in detail by Gaver and Lewis (1980), which also includes earlier references to related models. We note that the same arguments could be applied to Gamma random variables with integer degrees of freedom.

The attractiveness of these models is that they are AR(1) models where the underlying process is always positive and hence, can be used to model equity or bond prices in finance. Our version is a slight extension of existing NEAR(1) models in that Theorem 5 will be consistent with a Markov process for the state process rather than an i.i.d one, as in the current NEAR(1) literature.

Furthermore, if z_{t-1} is Markovian with transition matrix *P* such that $\pi = P\,\pi$, then:

$$\pi_j = \sum_{m=1}^{k} \pi_m P_{jm},$$

And

$$\phi(u) = \sum_{j=1}^{k} \exp(u\alpha_j)\, \phi(\beta_j u)\, \psi(\sigma_j u)\, \sum_{m=1}^{k} P_{jm}\, \pi_m$$

Alternatively,

$$\phi(u) =< \exp(u\alpha)\, \phi(\beta u)\, \psi(\sigma u), P\pi > \qquad (6)$$

There are a number of observations relevant to (4) and (6) as we theorise below:

Theorem 2:

Since $\pi = P\pi$ has multiple solutions for P, given π ,these different P's do not change the solution to equation (6). As an example, for k = 2, suppose $\pi = 0.5$, it then follows that $P_{11} = P_{22}$ but if $P_{11} = .2\ or\ .8$ in this context, the steady state distribution will be unaffected except through the change in position.

Thus, as is the case in this example, the steady-state values are equal (i.e 0.5) then such changes in the structure of the transition matrix should not influence steady-state values. Note, however, that this does not say anything about the speed at which the two processes in this example converge to the steady state. For more on speed of convergence, refer to Rosenthal, (1995). Since the processes converge to the steady state through different paths, simulating the steady state become a non-trivial procedure as explained in Section 3.

Theorem 3:

Suppose that in equation (6), α is zero, and $\phi(u)$ and $\psi(u)$ are infinitely differentiable moment generating functions and that the variance of the error process is constant.

Theorem 3.1.: If $\phi(u)$ is symmetric then $\psi(u)$ is symmetric; the proof is trivial.

Theorem 3.2: if $\psi(u)$ is symmetric then $\phi(u)$ is symmetric, proof by induction on Taylors series terms. We shall prove that all odd moments are zero:

Proof. $\phi(u) = (\sum_{j=1}^{k} \phi(\beta_j u)\ \pi_j)\ \psi(u)$

The coefficient of u^n for $\phi(u)$,

$$\phi_n = \sum_{j=1}^{k} \pi_j \sum_{s=0}^{n} \phi_{n-s} \beta_j^{n-s} \psi_s \qquad (7)$$

It follows that $\psi_1 = 0$ implies that $\phi_1 = 0$. We now suppose that $\psi_{2j+1} = 0$ implies that $\phi_{2j+1} = 0$ for j=0,…,k and consider ϕ_{2j+3}.From (7) and the inductive hypothesis and the properties of products of odd and even numbers, the result follows.

Theorem 4:

Suppose that $\alpha_j = 0$,the variance of the error term is constant and that we treat

$$\phi(u) = (\sum_{j=1}^{k} \phi(\beta_j u) \pi_j)\ \psi(u) \qquad (8)$$

as a statement about characteristic functions; then if at least one of the β_j is greater than 1 and all of them are non-negative, then for some n, the nth moment will not exist. The proof follows from using (7) again and noting that ϕ_n(the nth differential of $\phi(u)$), which is proportional to the nth moment (if it exists) can be expressed as:

$$\phi_n\left(1 - \sum_{j=1}^{k} \pi_j \beta_j^n\right) = \sum_{j=1}^{k} \pi_j \sum_{s=1}^{n} \phi_{n-s} \beta_j^{n-s} \psi_s$$

The requirement for the existence of ϕ_n is that $\sum_{j=1}^{k} \pi_j \beta_j^n$<1, which cannot hold for a large enough *n* under the assumptions of theorem 4. This result links local explosivity to fat tails. Thus, processes with locally explosive states will cease to have moments after some point. Ahmed et al (2018) derive similar conditions for the existence of a mean and variance for a TAR(1) process with an independently distributed exogenous trigger for state switching. We have generalized the result for the nth moment and for an exogenous trigger that is Markovian.

Theorem 5:

Assume that $\alpha_j = 0,$ and the variance of the error term is constant i.e. $\sigma_1 = \sigma_2 = \sigma$.

Consider now the special case, $k = 2, \beta_1 = 1, and\ \beta_2 = 0$. This is an important special case as it gives us a random walk in one regime and white noise in the other. Substituting into (8), we see that

$$\phi(u) = (\phi(u)\pi + 1 - \pi)\psi(u)$$

This can be re-arranged to yield

$$\phi(u) = \frac{(1-\pi)\psi(u)}{(1-\pi\psi(u))}$$

Since $|\psi(u)| \leq 1,\ \pi|\psi(u)|<1$ and $\phi(u)$ can be represented in terms of a valid series expansion which can be analysed term by term. Indeed,

$$\phi(u) = (1-\pi)\sum_{j=0}^{\infty}\pi^j\psi(u)^{j+1} \qquad (9)$$

The right-hand side is uniformly and absolutely convergent as a consequence of the Weirstrass M- test and thus we can integrate term by term.

We can now consider different choices for $\psi(u)$.

Suppose we have a normally distributed error term with mean zero and variance σ^2 i.e., $\psi(u) = \exp(-\frac{\sigma^2u^2}{2})$. Then $\psi(u)^{j+1}$ represents a normal random variable with mean 0 and variance (j+1) σ^2. We can identify the distribution of p_t as an infinite weighted sum of normal random variables of increasing variances but whose relative importance declines with a power of π. This process was analysed in Knight and Satchell (2013) and extended in Grynkiv and Stentoft (2018).

Likewise, assume the variance of the error is constant. If we consider a mean corrected Poisson so that $\psi(u) = \exp(\theta(\exp(iu) - 1 - iu))$ with mean parameter θ, we can identify the distribution of p_t as an infinite weighted sum of Poisson random variables of increasing means (j+1) θ but whose relative importance declines with a power of π.

This case can be extended to include intercepts, in which case,

$$\phi(u) = (\exp(i\alpha_1 u)\phi(u)\pi + (\exp(i\alpha_2 u)\,(1-\pi))\psi(u)$$

$$\phi(u) = \frac{\exp(i\alpha_2 u)\,(1-\pi)\psi(u)}{1-\exp(i\alpha_1 u)\pi\psi(u)}$$

Since $|\exp(i\alpha_1 u)\pi\psi(u)| < 1$, the analysis proceeds as before and

$$\phi(u) = (1-\pi)\exp(i\alpha_2 u)\sum_{j=0}^{\infty} \pi^j \psi(u)^{j+1} \exp(ij\alpha_1 u)$$

And we see that the jth component is as above but has a mean augmented by $j\alpha_1 + \alpha_2$.

The results can be extended to Vector Threshold Autoregressions as we show in the sub-section below.

Results for Vector Threshold Autoregressions

Because of the relevance of state variables influencing the conditional mean, we consider a vector autoregressive version of the model. We restrict ourselves to VTAR(1) to preserve the Markov properties.

$$p_t = \psi_{t-1} + \phi_{t-1} p_{t-1} + \sigma_{t-1}\eta_t$$

Let,

$$\psi_{t-1} =< \alpha, z_{t-1} > \qquad \phi_{t-1} =< \beta, z_{t-1} > \qquad \sigma_{t-1} =< \sigma, z_{t-1} >$$

As before but now p_t and η_t are k by 1 vectors.

$$\alpha = (\alpha_1, \alpha_2, \alpha_3)' \quad \beta = (\beta_1, \beta_2, \beta_3)' \quad \sigma = (\sigma_1, \sigma_2, \sigma_3)' \quad \eta_t \sim iid(0,1)$$

$z_t \in \{e_1, e_2, e_3\}$ $\quad e_1 = \begin{pmatrix} I_k \\ 0 \\ 0 \end{pmatrix}$ $\quad e_2 = \begin{pmatrix} 0 \\ I_k \\ 0 \end{pmatrix}$ $\quad e_3 = \begin{pmatrix} 0 \\ 0 \\ I_k \end{pmatrix}$ where $\alpha_i, \beta_i, \sigma_i$ are k by k matrices. The stationarity of such systems is discussed in Grynkiv and Stentoft (2018). Following exactly the same steps as in our previous derivation,

$$\phi_t(u) = \sum_{j=1}^{k} \pi_j \left(\exp[u' < \alpha, e_j >]\, \phi_{t-1}[u' < \beta, e_j >] \psi(u' < \sigma, e_j >)\right) \quad (10)$$

where u is now a k × 1 vector; $\phi_t(u)$ is now the joint moment generating function or characteristic function of the vector p_t with similar changes in definition applying to other terms.

Similarly, the steady state relationship will become

$$\phi(u) = \sum_{j=1}^{k} exp\left(u'\alpha_j\right) \phi\left(u'\beta_j\right) \pi_j \, \psi(u'\sigma_j) \qquad (11)$$

Where, again the terms in (11) refer to joint moment generating functions or characteristic functions. We see that (10) can be used to compute conditional moments, conditional upon both time and state variables.

In the next section, we present some numerical investigations where we look at the impact of local explosivity on the properties of the steady-state.

3. A caveat on simulating the steady state:

Simulating the steady state for processes similar to the ones considered in section 2 is not as straight forward as it may appear at first glance and warrants further consideration, which this section seeks to provide. Generating a discrete Markov chain, essentially a variable that takes discrete integer values, 0 or 1 depending on the transition matrix P, does not generate a steady state Markov chain, but rather a path to the steady state. It is common in this literature to simulate steady state paths, rather than a steady state Markov Chain. Indeed, in our earlier work, we work with paths and not steady states as does Timmerman in his article (see Figures 1-6 in his articles for example). While simulating steady state paths sufficed for our earlier work, we need to simulate the steady state in order to corroborate our results from section 2. Otherwise, the underlying moments of the simulated series can be different even if they converge to the same steady state.

This path is dependent upon the transition matrix P; if states are persistent, as determined by the transition probabilities of the process staying in the prevailing state (p_{ii}), this path may diverge significantly from the steady state. On the opposite spectrum is a transition matrix with frequent state switches, due to higher switching probabilities, which will take a different path to the steady state. Although the steady-state probabilities of both paths are identical, the dynamics vary due to the different paths taken by the processes.

We need to consider how discrete Markov Chains converge to their steady states. The usual definition is based on total variation distance and considers the supremum, taken over measurable subsets A, of the absolute difference between $v(A)$ and $u(A)$ where $v()$ and $u()$ are the two probability measures, see Rosenthal (p. 389, 1995). Whilst our process will converge in this sense, it will not converge almost surely along a sample path; intuitively, it keeps moving from state to state.

We illustrate this by considering two different transition matrices that correspond to the steady-state probability vector $\begin{bmatrix}0.5\\0.5\end{bmatrix}$. We consider a transition matrix with highly persistent states $P_1 = \begin{pmatrix}0.9 & 0.1\\0.1 & 0.9\end{pmatrix}$ and a transition matrix with frequent state switches $P_2 = \begin{pmatrix}0.5 & 0.5\\0.5 & 0.5\end{pmatrix}$. While both processes approach the same steady states, the simulated series have different probability distributions. Specifically, persistence of the non-stationary process, corresponding to $\beta_j > 1$, causes the path to diverge far from the steady-state values, resulting in a process that has extreme values with significant probability, which also obtains very high kurtosis. With frequent state-switches, the simulated series comes closer to a normally distributed process.

To remedy this, each of the above paths is simulated for 10,000 periods and the process is repeated 2000 times; the parameter values in the two switching states are 0 in state 1 and 1 in state 2. We also carry out simulations when the switching states correspond to values of 0.1 in state 1 and 1.1 in state 2. We record the average of the first 4 moments of both paths in table 1 above. As mentioned previously, the persistent path obtains a much higher kurtosis and standard deviation than the more frequently switching path, even though both paths continue to be symmetrically distributed. An alternative approach would be to write out the solution to equation (1) and simulate directly by taking long samples of the error term and the exogenous process. From Table 1, we observe that the highest 2nd and 4th moments are obtained for the persistent state when $\beta_1 = 0.1, \beta_2 = 1.1$. This is despite the fact that this process also converges to a steady state vector [0.50 0.50]. The moments are significantly different for the alternative paths corresponding to P_2.

Table 1

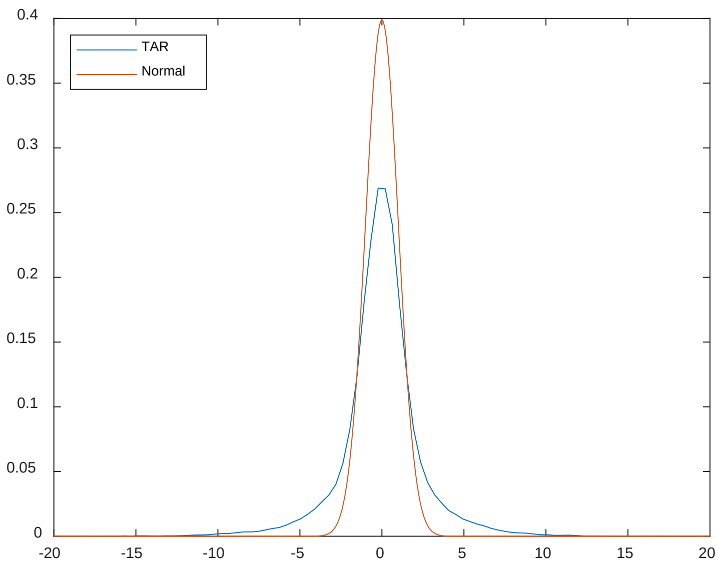
Steady State vector	Transition Matrix	$\boldsymbol{\delta}$	Mean	Stdev	Skewness	Kurtosis
[0.50 0.50]	$P_1 = \begin{pmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{pmatrix}$	0	0.0004	2.450	0.00289	8.786
[0.50 0.50]	$P_1 = \begin{pmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{pmatrix}$	0.10	-0.170	78.17	0.2761	424.7
[0.50 0.50]	$P_2 = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}.$	0	-0.0001	1.4139	-0.0004	4.480
[0.50 0.50]	$P_2 = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}.$	0.10	0.0000	1.6014	-0.0012	6.837

Table 1 reports the first four moments of simulated Threshold Autoregressive series with Markov-switching exogenous variables. Each series is 10,000 observations long and 2000 series were simulated for each set of values. P represents the transition matrix and delta represents a parameter that determines the values of β_j in the two states; $\beta_1 = \delta, \beta_2 = 1 + \delta$.

Figures 1 and 2 below further highlight the different distributions that result from the different paths along with cumulative probabilities corresponding to normal probability density function's quintile values. Tail probabilities are much higher for the more persistent path, which also has more extreme values, especially when the non-stationary state becomes explosive. Tail probabilities are 28.7% and 30.8% respectively in Figures 1-1 and Figures 1-2 which correspond to the more persistent transition matrix, P_1. In fact, if we consider Figure 1-2, we are unable to observe the distribution corresponding to $\beta_1 = 0.1, \beta_2 = 1.1$. Figure 1-3 considers this distribution by plotting the distribution over a wider range. This distribution appears symmetrical but also obtains positive probability for extreme values.

Figure 1-1

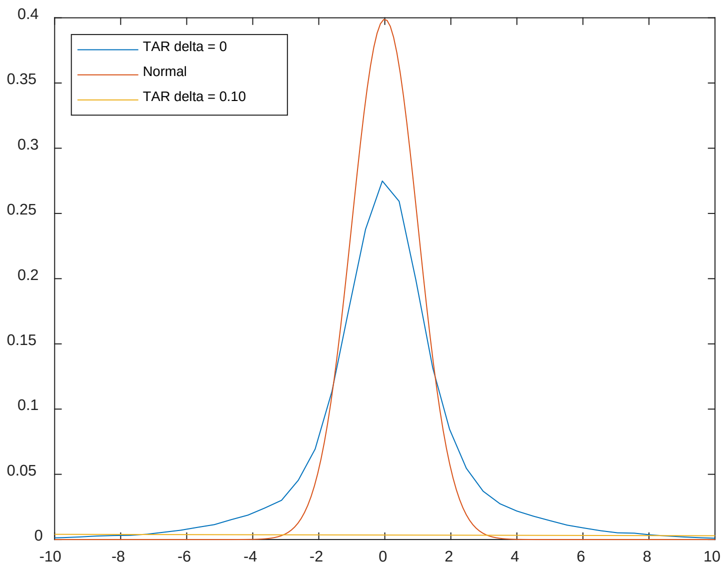
$$P_1 = \begin{pmatrix} 0.90 & 0.10 \\ 0.10 & 0.90 \end{pmatrix}, \pi = \begin{bmatrix} 0.50 \\ 0.50 \end{bmatrix}, \delta = 0, \beta_1 = \delta, \beta_2 = 1 + \delta$$



Quintile1	Quintile2	Quintile3	Quintile4	Quintile5
28.72%	14.35%	13.89%	14.36%	28.69%

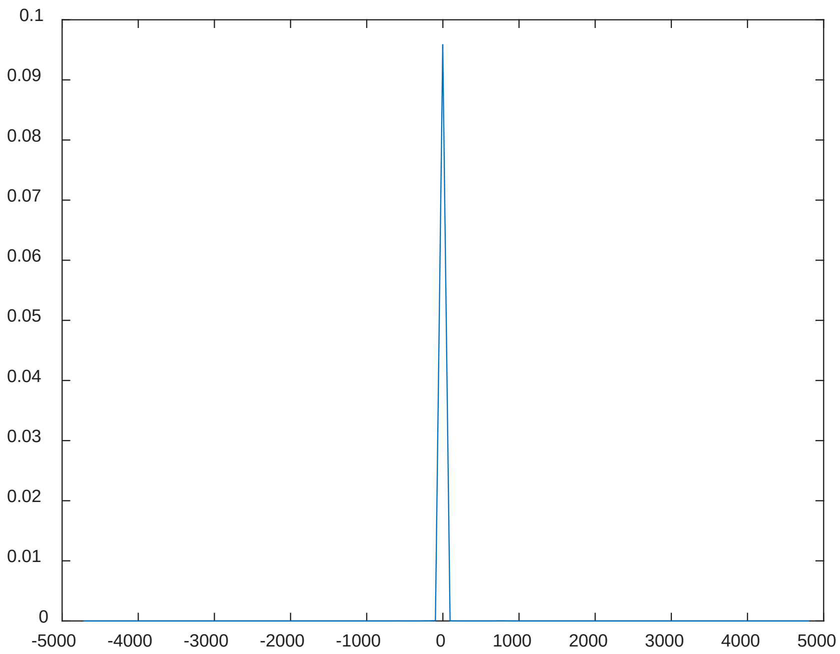
Figure 1-2

$$P_1 = \begin{pmatrix} 0.90 & 0.10 \\ 0.10 & 0.90 \end{pmatrix}, \pi = \begin{bmatrix} 0.50 \\ 0.50 \end{bmatrix}, \delta = 0.10, \beta_1 = \delta, \beta_2 = 1 + \delta$$



Quintile1	Quintile2	Quintile3	Quintile4	Quintile5
30.83%	12.92%	12.53%	12.92%	30.8%

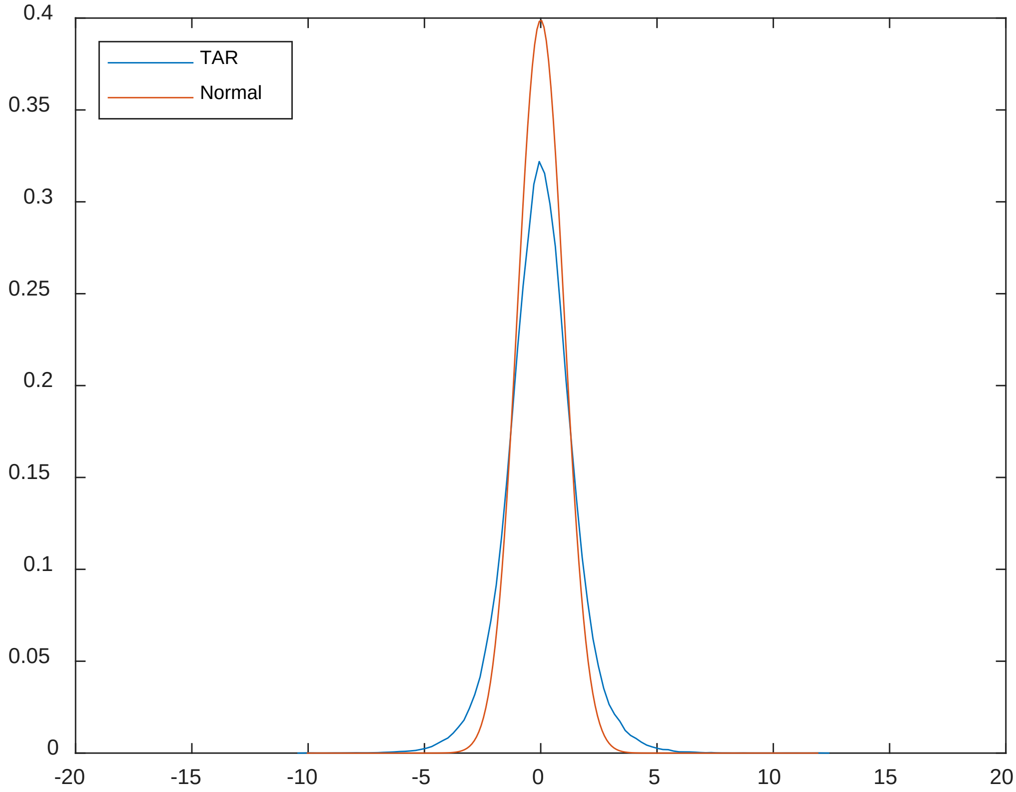
Figure 1-3



The distributions of simulated series corresponding to the more frequently switching path (transition matrix P_2) on the other hand, have lower tail probabilities and distributions that are closer to the standard normal distribution. Figure 2-1 contrasts the distribution for the series corresponding to path P_2 to a standard normal. While this simulated distribution has heavier tails, as evident from the higher probabilities corresponding normal quintiles, its 2[nd] and 4[th] moments are much closer to the normal than to the simulated series for path P_1. Similarly, the distribution in Figure 2-2, corresponding to $\beta_1 = 0.10, \beta_2 = 1.10$ and path P_2 looks much closer to a normal distribution than to its P_1 counterpart.

Figure 2-1

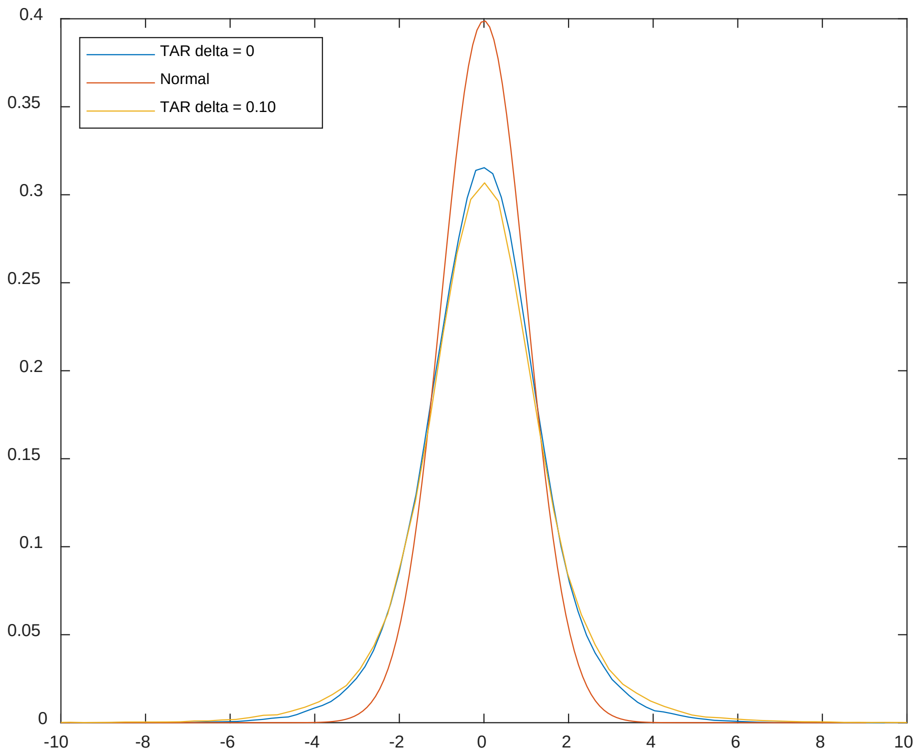
$$P_2 = \begin{pmatrix} 0.50 & 0.50 \\ 0.50 & 0.50 \end{pmatrix}, \pi = \begin{bmatrix} 0.50 \\ 0.50 \end{bmatrix}, \delta = 0, \beta_1 = \delta, \beta_2 = 1 + \delta$$



Quintile1	Quintile2	Quintile3	Quintile4	Quintile5
25.19%	16.73%	16.16%	16.73%	25.19%

Figure 2-2

$$P_2 = \begin{pmatrix} 0.50 & 0.50 \\ 0.50 & 0.50 \end{pmatrix}, \pi = \begin{bmatrix} 0.50 \\ 0.50 \end{bmatrix}, \delta = 0.10, \beta_1 = \delta, \beta_2 = 1 + \delta$$



Quintile1	Quintile2	Quintile3	Quintile4	Quintile5
26.17%	16.08%	15.49%	16.08%	26.17%

Thus, we need a different approach to generate the steady-state distribution of the threshold autoregressive process with Markovian triggers that are independent of the transition matrix, subject to the same steady state. It is important to understand that the results derived in the section above, correspond to the steady state itself and not to the path of the process tending to a steady state, which as we have shown in this section, depends on the transition matrix. We describe how we simulate the steady state distribution in the next section.

4. Simulation results

In order to simulate a distribution for the steady-state we observe the Markov chain 10,000 times, however, each observation is 1000 time periods or steps apart. Thus, the Markov chain we simulate is 10,000,000 periods long and the steady state simulation is 10,000 observations in length. We verified that the each series simulated this way converges to the steady-state probability vector while at the same time being independent of the transition matrix, P. We did this by simulating steady states for different transition matrices P but that shared the same steady states.[2] The steady-state chain is then used to simulate the following threshold model:

$$y_t = \alpha + \beta_{it} y_{t-1} + \eta_t$$

Where β_{it} depends on the value taken by the exogenous discrete Markov state variable z_i. Our simulations assume that $\eta_t \sim N(0,1)$ and that $\alpha = 0$. In order for our simulated series to have a steady-state stationary distribution, we require that the criterion $\sum_{i=1}^{2} \pi_i \ln|\beta_i| < 0$ be satisfied. Since we are considering a two-state process, the criterion can alternatively be written as $\beta_1^{\pi} \beta_2^{1-\pi} < 1$. The criterion is trivially satisfied when $\beta_1 = 0$ as $\ln(0) = -\infty$. When $\beta_1 = \delta$ and $\beta_2 = (1+\delta)$, the criterion becomes, $\delta^{\pi}(1+\delta)^{1-\pi} < 1$. Note, that the expression, $\delta^{\pi}(1+\delta)^{1-\pi}$ is maximized for $\pi = 0$, when it is greater than one and the critical point for π is when $\pi = \frac{-\ln(1+\delta)}{\ln(1+\delta)-\ln(\delta)}$.

We consider a maximum δ of 0.10,and check that the criterion is satisfied for all our simulations and that a steady state distribution does exist. For exposition, we have included the value taken by the criterion function for each set of simulations in column 7 of table 2 below.

Steady state distributions obtained this way can be analysed through the results derived in section 2. For each set of steady state vectors in table 2, we simulated threshold autoregressive series (as described above) 2000 times; the parameter values in the two switching states are δ in state 1 and $1+\delta$ in state 2. Thus, the first state is always stationary while the second state is either a random walk or explosive.

Some patterns are exhibited clearly. The distributions of the series appear to be centred on zero and statistically their skewness (column 5) is not significantly different from zero; this follows from Theorem 3. Since the error term has an even distribution it follows that the distribution of our simulated series will also be even, symmetric and centred on zero. i.e. $\psi(u) = \psi(-u)$ implies that $\phi(u) = \phi(-u)$.

The standard deviation (column 2), however, does appear to be much larger than the standard deviation of the error process driving the threshold process. The series have excess kurtosis (column 6), which should not come as a surprise since the series display non-stationary behaviour when $\beta_i \geq 1$; this state leads to excess kurtosis and the higher standard deviation. As we deviate from our base case, ($\delta = 0$), we note a clear pattern in the 2nd and 4th moments of the series. Both the standard deviation and kurtosis start to increase; again, since this behaviour is caused by the non-stationary state becoming increasingly more explosive ($\beta_i = 1+\delta, when\ \delta > 0$). The pattern is repeated irrespective of the steady state vector chosen.

[2] These results have not been included but are available upon request.

Table 2

Steady State vector $[\boldsymbol{\pi}\ \ \mathbf{1}-\boldsymbol{\pi}]$	$\boldsymbol{\delta}$	**Mean**	**Stdev**	**Skewness**	**Kurtosis**	**Criteria** = $\sum_{i=1}^{2}\boldsymbol{\pi}_i \ln\lvert\boldsymbol{\beta}_i\rvert$
[0.90 0.10]	0	0.0001	1.0544	-0.0012	3.306	$-\infty$
[0.90 0.10]	0.01	-0.0007	1.0552	0.0004	3.311	-4.1437
[0.90 0.10]	0.03	0.0001	1.0583	-0.0008	3.336	-3.1529
[0.90 0.10]	0.05	0.0007	1.0608	-0.0006	3.362	-2.6913
[0.90 0.10]	0.10	0.0007	1.0714	-0.0007	3.442	-2.0628
[0.80 0.20]	0	0.0004	1.118	-0.002	3.597	$-\infty$
[0.80 0.20]	0.01	-0.0000	1.121	-0.0007	3.630	-3.6821
[0.80 0.20]	0.03	0.0001	1.128	-0.0016	3.705	-2.7993
[0.80 0.20]	0.05	0.0005	1.134	0.003	3.762	-2.3868
[0.80 0.20]	0.10	0.0001	1.152	0.0003	3.967	-1.8230
[0.60 0.40]	0	0.0004	1.291	-0.003	4.195	$-\infty$
[0.60 0.40]	0.01	-0.0011	1.300	-0.001	4.277	-2.7591
[0.60 0.40]	0.03	-0.0001	1.316	-0.005	4.421	-2.0921
[0.60 0.40]	0.05	-0.0007	1.339	-0.002	4.660	-1.7779
[0.60 0.40]	0.10	-0.0004	1.400	0.002	5.410	-1.3434
[0.50 0.50]	0	0.0017	1.413	0.0026	4.470	$-\infty$
[0.50 0.50]	0.01	-0.0004	1.428	-0.0042	4.612	-2.2976
[0.50 0.50]	0.03	-0.0006	1.460	-0.0001	4.915	-1.7385
[0.50 0.50]	0.05	-0.0004	1.494	-0.0021	5.302	-1.4735
[0.50 0.50]	0.10	0.001	1.601	0.0227	6.720	-1.1036

Table 2 reports the first 4 moments of the simulated series along with the stationary criterion. Each row corresponds to 2000 threshold autoregressive simulations with Markov triggers, 10,000 observations long; columns 3-6 report average moments.

Unsurprisingly, the higher the steady state probability of the stationary state ($\beta_i = \delta$), the closer the process's distribution is to a normal distribution. This is reflected through the first four moments. For instance, when the stationary state occurs 90% of the time (as in rows 2-6 of Table 2), the standard deviation and kurtosis are both lower compared to corresponding cases (i.e. same δ) when the stationary state occurs less than 90% of the time. When $\delta = 0.5$, the standard deviation and kurtosis respectively are 1.061 and 3.362 for $\pi = 0.90$, 1.134 and 3.762 for $\pi = 0.80$, 1.339 and 4.660 for $\pi = 0.60$ and 1.494 and 5.302 for $\pi = 0.5$. While all distributions appear symmetric and centred on 0, they become increasingly leptokurtic as π falls.

We also plot sample distributions for $\delta = 0$ and $\delta = 0.10$ for each set of steady state vectors (Figures 3-6) and carry out quintile analysis by calculating the weights in the distribution corresponding to the quintile values of a normal distribution i.e. we find $P(y_t) < q_1, q_1 < P(y_t) < q_2, q_2 < P(y_t) < q_3, q_3 < P(y_t) < q_4, P(y_t) > q_4$, where q_i correspond to the normal distribution's quintile values. We note that tail probabilities i.e. those corresponding to the 1st and 5th normal quintile values, go up as the steady state probabilities for the non-stationary state go up. They are also dependant on the value of δ and as we increase explosivitiy in the non-stationary state (by increasing δ), tail probabilities increase and the distributions moves farther away from a normal. The symmetry of the distribution is also reflected in these probabilities.

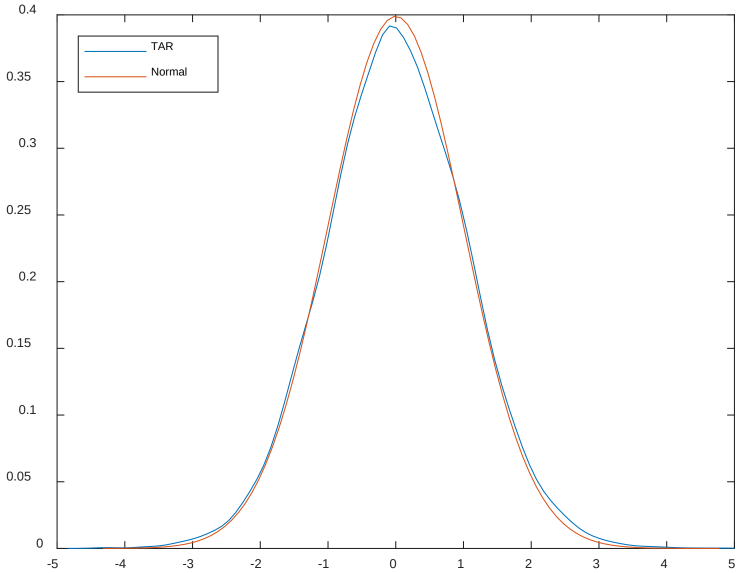
The results above depend only on the steady state vector $\begin{bmatrix} \pi \\ 1-\pi \end{bmatrix}$ and not on the transition matrix (the dynamic path) that generates this steady state vector. We note that all series generated this way have excess kurtosis due to the presence of a non-stationary state.[3] Below we simulate pdfs for some of the cases considered in Table 2. The graphs only report the distribution for $\delta = 0$ and $\delta = 0.10$ so that we can analyse how far the distribution moves from a normal as we increase δ and increase the probability of the non-stationary part of the distribution. We also draw a comparison with a normal distribution for illustrative purposes.

From Figures 3-1 and 3-2 we can see that when the stationary state is dominant $(\pi = 0.90)$, the distribution appears very close to a normal distribution even if δ is increased from 0 to 0.10. Contrast this distribution with Figures 6-1 and 6-2 where the stationary and non-stationary state occur with equal probability. The distributions in Figures 6-1 and 6-2 are significantly leptokurtic, with tail probabilities of 25.2% and 26.2% respectively as opposed to tail probabilities of 20.8% and 21% in Figures 3-1 and 3-2.

These results are interesting particularly for those relying on DSGE models. In finance, asset prices often exhibit locally non-stationary behaviour, which leads to leptokurtosis in the series. Similarly, Macroeconomists often consider different shock mechanisms in DSGE models. The results in this article will assist Macroeconomists in considering shock processes that follow threshold auto regressions with a non-stationary state. Since we have outlined a procedure for deriving analytical expressions, this would enable Macroeconomists to analyse locally explosive shock processes which nevertheless are stationary overall and facilitate the implementation of numerical solutions.

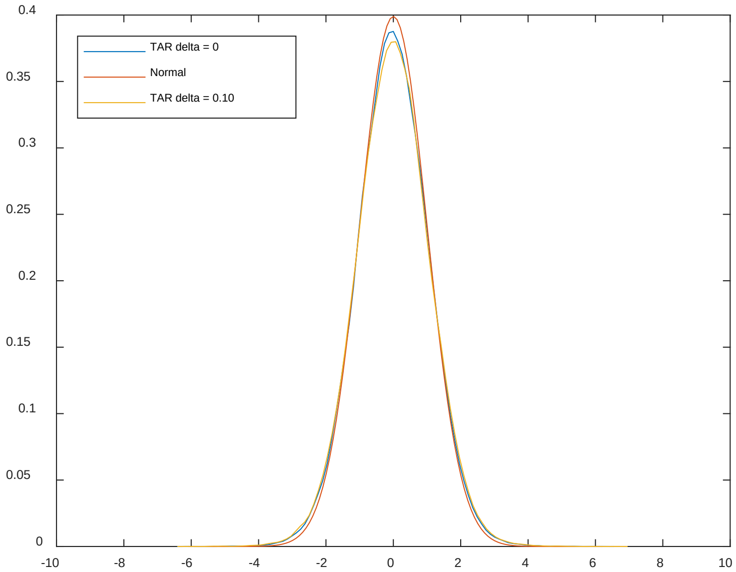
[3] We have checked that the kurtosis exist by verifying that $1 - \sum_{j=1}^{2} \pi_j \, \beta_j^4 < 0$. Indeed, for our most extreme case $(\pi_1 = 0.5, pi_2 = 0.5, \beta_1 = 0.10, \beta_2 = 1.1)$, the kurtosis does exist as the criterion for its existence is satisfied: $1 - \sum_{j=1}^{2} \pi_j \, \beta_j^4 = 0.7321$.

FIGURE 3-1: $P = \begin{pmatrix} 0.95 & 0.05 \\ 0.45 & 0.55 \end{pmatrix}, \pi = \begin{bmatrix} 0.90 \\ 0.10 \end{bmatrix}, \delta = 0, \beta_1 = \delta, \beta_2 = 1 + \delta$



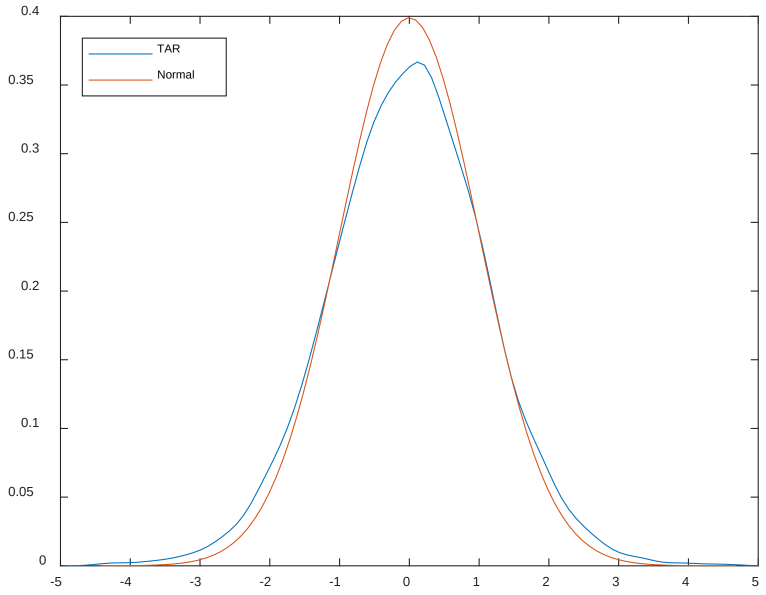
Quintile1	Quintile2	Quintile3	Quintile4	Quintile5
20.81%	19.47%	19.41%	19.53%	20.78%

FIGURE 3-2: $P = \begin{pmatrix} 0.95 & 0.05 \\ 0.45 & 0.55 \end{pmatrix}, \pi = \begin{bmatrix} 0.90 \\ 0.10 \end{bmatrix}, \delta = 0.10, \beta_1 = \delta, \beta_2 = 1 + \delta$



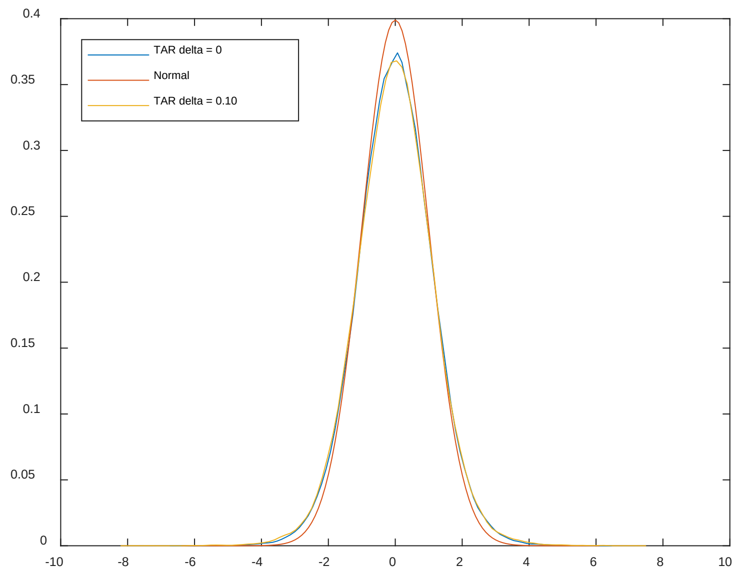
Quintile1	Quintile2	Quintile3	Quintile4	Quintile5
21.00%	19.35%	19.23%	19.38%	21.03%

FIGURE 4-1: $P = \begin{pmatrix} 0.95 & 0.05 \\ 0.20 & 0.80 \end{pmatrix}, \pi = \begin{bmatrix} 0.80 \\ 0.20 \end{bmatrix}, \delta = 0, \beta_1 = \delta, \beta_2 = 1 + \delta$



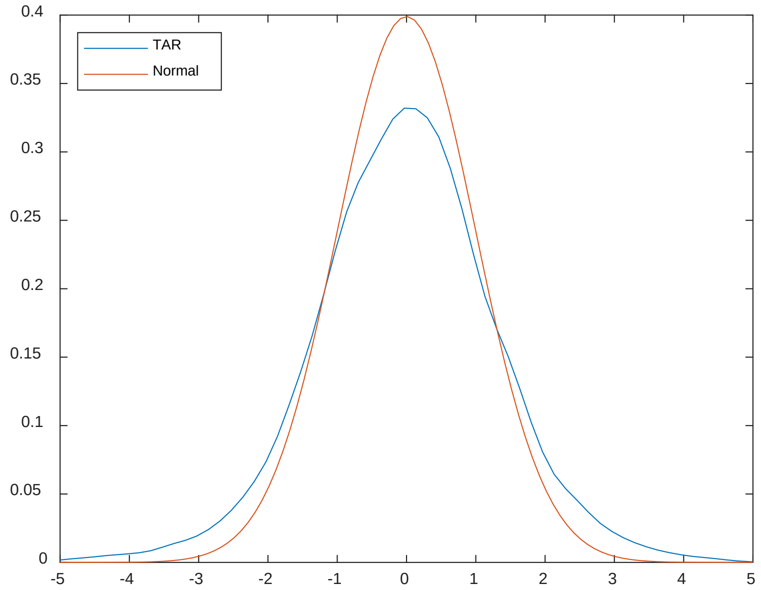
Quintile1	Quintile2	Quintile3	Quintile4	Quintile5
21.67%	18.95%	18.73%	18.94%	21.71%

FIGURE 4-2: $P = \begin{pmatrix} 0.95 & 0.05 \\ 0.20 & 0.80 \end{pmatrix}, \pi = \begin{bmatrix} 0.80 \\ 0.20 \end{bmatrix}, \delta = 0.10, \beta_1 = \delta, \beta_2 = 1 + \delta$



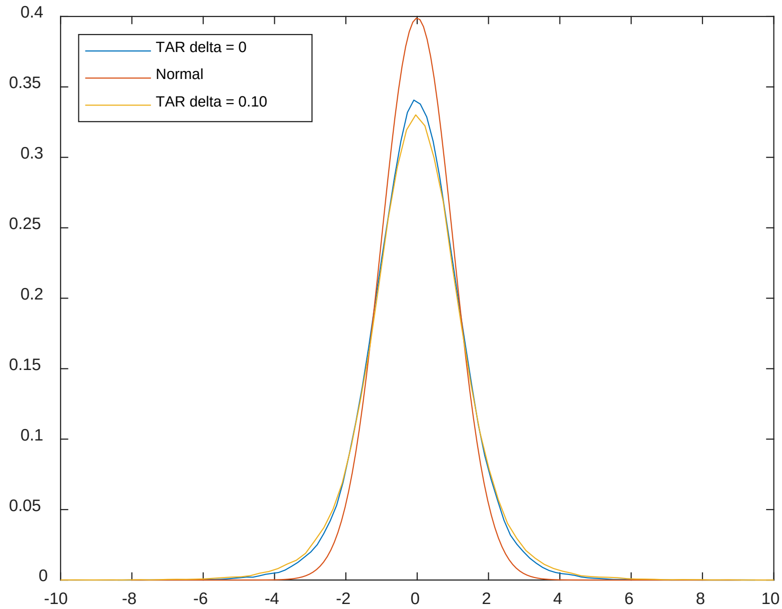
Quintile1	Quintile2	Quintile3	Quintile4	Quintile5
22.06%	18.70%	18.46%	18.70%	22.08%

FIGURE 5-1: $P = \begin{pmatrix} 0.90 & 0.10 \\ 0.15 & 0.85 \end{pmatrix}, \pi = \begin{bmatrix} 0.60 \\ 0.40 \end{bmatrix}, \delta = 0, \beta_1 = \delta, \beta_2 = 1 + \delta$



Quintile1	Quintile2	Quintile3	Quintile4	Quintile5
23.84%	17.58%	17.12%	17.61%	23.85%

FIGURE 5-2: $P = \begin{pmatrix} 0.90 & 0.10 \\ 0.15 & 0.85 \end{pmatrix}, \pi = \begin{bmatrix} 0.60 \\ 0.40 \end{bmatrix}, \delta = 0.10, \beta_1 = \delta, \beta_2 = 1 + \delta$



Quintile1	Quintile2	Quintile3	Quintile4	Quintile5
25.58%	17.11%	16.63%	17.11%	24.56%

FIGURE 6-1: $P = \begin{pmatrix} 0.50 & 0.50 \\ 0.50 & 0.50 \end{pmatrix}, \pi = \begin{bmatrix} 0.50 \\ 0.50 \end{bmatrix}, \delta = 0, \beta_1 = \delta, \beta_2 = 1 + \delta$

Quintile1	Quintile2	Quintile3	Quintile4	Quintile5
25.16	16.73	16.16	16.74	25.21

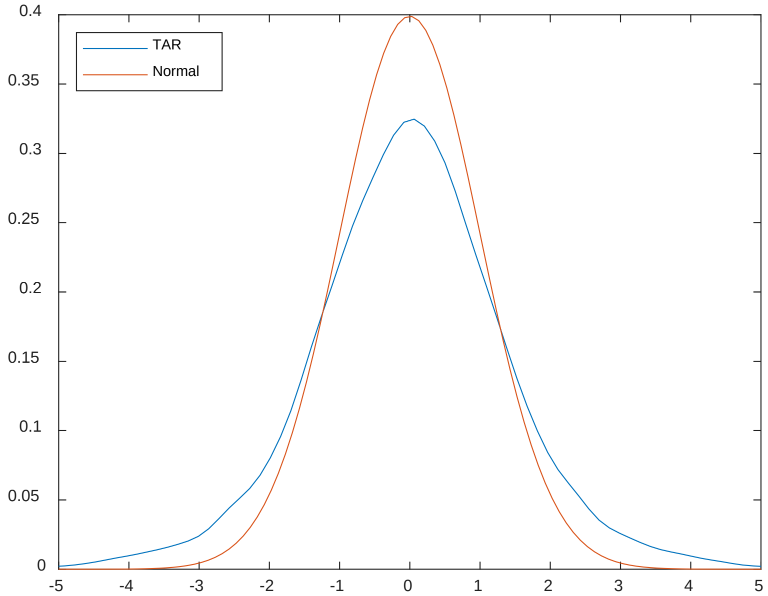
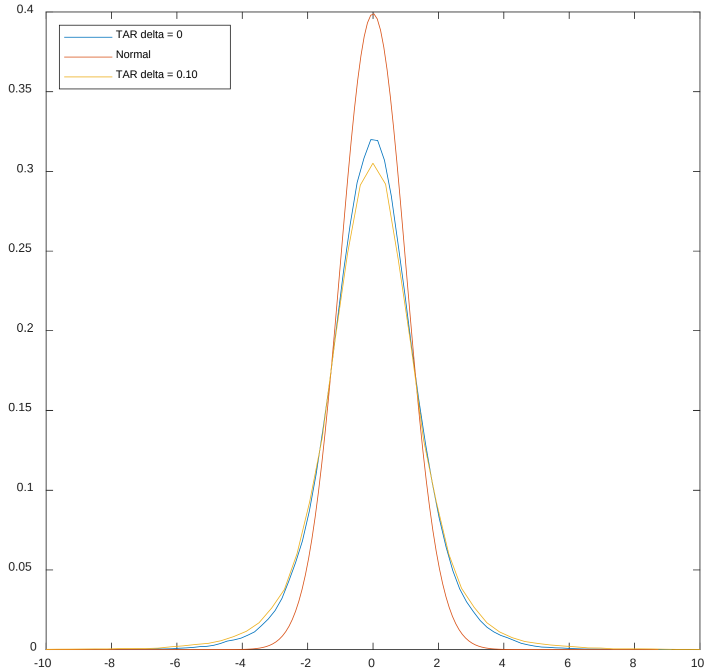


FIGURE 6-2: $P = \begin{pmatrix} 0.50 & 0.50 \\ 0.50 & 0.50 \end{pmatrix}, \pi = \begin{bmatrix} 0.50 \\ 0.50 \end{bmatrix}, \delta = 0.10, \beta_1 = \delta, \beta_2 = 1 + \delta$



Quintile1	Quintile2	Quintile3	Quintile4	Quintile5
26.17	16.10	15.48	16.10	26.17

5. Conclusion

In this article, we have derived formulae for characteristic functions for Threshold Autoregressive models of order 1, which have a Markovian state-switching variable. In doing so, we have not only improved the results first considered in Timmerman (2000) but have generalized the formulae to a great degree. These formulae will allow readers, if they are so inclined, to derive analytical moments for TAR Models in this class for a range of different error specifications. We believe that these will have applications in both finance and applied macroeconomics.

Considering a special case of interest, we have also shown, using simulation analysis that the existence of a non-stationary state in a TAR model, can cause the distribution of a particular series to deviate significantly from normality. The further away the non-stationary state moves from a random walk, the farther away the distribution is from that of a normal. Models for asset prices often consider a mixture of stationary and non-stationary states and we believe that this simulation analysis will aid researchers in better understanding the behaviour of asset prices that go through locally explosive states. In Sections 1 and 2 we have further outlined how our results may be applicable to applied macroeconomic and finance literatures.

Thus, our article makes significant contributions to the existing literature on TAR models and also offers insights into how such models may be put to use in applied economics.

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