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GAMES ON MULTI-LAYER NETWORKS

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A set of agents is connected by two distinct networks, with each network describing access to a different local public good. Agents choose which networks to invest in, and neighbouring agents' investments in the same good are strategic substitutes, as are an agent's two investment choices. There are always equilibria where any investing agent bears all local investment costs and others free-ride. When investment in one good reduces marginal benefit from investment in the other, agents free-riding in one good may invest more profitably in the other, and equilibrium payoffs are more evenly distributed. This need not reduce aggregate payoff.

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Abstract

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1 Introduction

At work, school, or in our communities, we will often have opportunity to share the work of others for our own benefit. We get to enjoy the sights and smells when our neighbours plant their gardens, and just one co-author's brilliant insight may be enough to push a group project forward. These are instances of non-excludable local public goods, where individual contributions are shared by anyone with access to them. Our research is inspired by the example of research and development between firms. Firms may invest to innovate new technologies, but technological breakthroughs may quickly be adopted by other firms, and firms may be incentivised to withhold investment in the hope that others will innovate first.

Firms may have multiple research opportunities and limited resources, and maximising profit requires allocating these resources to where they are most efficient. In our model, return on investment has two factors: declining returns when multiple connected firms research the same technology, and increased costs when a firm spreads its resources widely across many technologies. Our work answers questions as to how firms, or any agents in networks, will best allocate their resources between networks.

Because firms have research links in multiple technologies, there are multiple overlapping networks in which they connect, and actions in each network are strategically determined by the linking structure and the investments of other firms in all networks. Multiple networks may be used to model many situations; for instance, as individuals we have networks of friends and networks of colleagues, with each relationship providing a different set of costs and rewards. Nearly all prior network literature involves agents in a single network; thus, we believe research connecting strategic decision making in multiple networks is novel, and will help to open new ways of thinking about how agents connect.

There is one set of agents with two public goods to invest in. Each good has a distinct set of connections describing the pairs of agents who share benefits, and benefits for any agent depend solely upon the total investment of all agents they share links with in a good. Because of the cost of investing, an agent will always prefer a neighbour's investment to an equal amount of their own investment. Firms will, when possible, seek to avoid investment when their neighbours are willing to bear the cost of investing instead.

Agents who make the choice to invest in both goods face an increased marginal cost relative to their investment in each good. This effect is labelled *distraction*, and measures inefficiency from spreading research efforts too widely. As distraction increases, firms are more heavily penalised for investing in both goods, which will incentivise them to select one good for investment. When there is no distraction, the investment decisions for each good are independent, and the model nests the work of Bramoullé and Kranton (2007).

Because marginal benefit decreases when neighbours invest, two linked agents are strategic substitutes. As well, because distraction decreases investment payoffs, an agent's two investments are strategic substitutes. We show how each of these factors affects equilibrium, and prove the existence in any network of equilibria where any investing agent receives no investment help from her neighbours.

Welfare is measured according to two factors: aggregate payoff and the distribution of payoffs. Owing to each agent's self-interested decision making, investment will always be less than an aggregate payoff maximising level, as the externalities from investment are always positive. We discuss how an agent's neighbours affect his choice of which good to invest in, which can factor into an agent selecting the wrong investment good with respect to aggregate welfare. We establish that agents who invest in both goods will always be the least well-off agents in any equilibrium.

Increasing the cost of investment will decrease the payoff generated by any agent who invests in both layers, when holding the investments of all agents fixed. However, strategic implications when agents act in response to an increase in distraction may provide benefit to these investors. As making two investments becomes unprofitable, investors are forced to choose a single good for investment. This raises minimum payoffs in a network, as dual-investors fair worst, and, so long as there are other investors to replace the lost investment, the efficiency of investments can rise. Adding links is often beneficial for aggregate payoff, as new links spread investment benefits, but some links may reduce aggregate payoff if they connect investors who respond by lowering their investments.

For an equilibrium to be stable—when a sequence of myopic best responses by all agents to a small perturbation of equilibrium converges to the original equilibrium—requires additional constraints on the connections of any non-investing agent. As well, these constraints are more strict when a non-investing agent is only linked to investors who invest in both goods. This leads to the conclusion that the subset of equilibria that are stable will be the most equitable equilibria, as they will contain a higher proportion of investors in one good versus those who invest in none or both.

We discuss an adaption of the model that, while simplifying payoff structure, allows for a much broader set of strategic interactions. On each layer, actions between agents may be strategic complements or substitutes, and this flexibility applies also to the relationship between an agent's two actions. In this simplified extension, we show the parameter space on which a unique equilibrium exists.

This chapter contributes to an extensive literature on public good investment, where public goods are generally undersupplied by voluntary contributions. Warr (1983) and Bergstrom et al. (1986) show that aggregate contribution and individual consumption are invariant to transfers between contributing agents, provided that transfers leave all contributing agents above their original level of private consumption. Network models allow for local public goods, where benefits are shared only by agents connected to contributors (e.g. Allouch 2013; Allouch 2015; Bramoullé and Kranton 2007). Elliott and Golub's (2019) model has one universal public good and a weighted network describing heterogeneous inter-agent benefits from contribution. Foster and Rosenweig (1995) empirically show that knowledge does spread through a network, but more slowly than it may if individuals were considering their neighbours. In contrast to these models, our model is novel because it has two public goods on two networks; each good is underinvested in, and an agent's choice of good may have positive or negative externalities for neighbours.

We contribute to the study of strategic decision-making in networks. In these games, the connections between agents determine the strategic effects that their actions have on each other. The following surveys: Bramoullé and Kranton (2016) and Jackson and Zenou (2015), and books: Goyal (2007) and Jackson (2008), provide a starting point for this literature. Linear quadratic payoff models are highly flexible, allowing for a wide range of strategic effects through the adjustment of a single parameter (e.g. Ballester, Calvó-Armengol et al. 2006; Ballester, Zenou et al. 2010; Calvó-Armengol and Zenou 2004). Bramoullé, Kranton and D'Amours's (2014) generic model nests many of these games, and they analyse what drives games of strategic substitutes to have corner solutions and multiple equilibria. Galeotti et al. (2010) find that when agents in games of strategic substitutes have incomplete network information and act upon their expected location in a network, unique equilibria exist. By connecting two networks in our model, we show that the outcome in one network determines the set of potential outcomes in the other.

Our model helps to explain a problem of allocating resources in a network. When agents have a fixed budget to invest they are unable to invest in all profitable opportunities and must find the set of investments that is most efficient (e.g. Baumann 2015; Bloch and Dutta 2009; Salonen 2014). In contrast, in our model an agent's budget is unconstrained, but the choice of taking multiple investments makes each investment less profitable. Thus, a profit optimising agent may be forced to choose between two network actions that are independently profitable because they are strategic substitutes.

We contribute to the understanding of how network links may create or reinforce inequality. Dalton (1920) and Atkinson (1970) explore which measures of dispersion in a population best capture inequality. In Gagnon and Goyal's (2017) model, agents have a network action and a market action. Taking the market action changes the network payoffs, and when poorer agents in the network receive greater benefit from the network action inequality falls. In our model, both actions are network actions, but costly actions in one network provide benefits in the other, which can reduce inequality.

Our model provides insight into the costs, benefits and strategies induced by R&D networks. In some models (e.g. Goyal, Konovalov et al. 2008; Goyal and Moraga-González 2001) cost reduction is greatest when firms cooperate, and prior to market competition research spending is complementary. In Westbrock (2010), links provide fixed R&D benefits. Our model follows most closely the assumptions of Bramoullé and Kranton (2007), that one firm's research may substitute for another firm's research, and because firms do not consider the benefits they provide to their neighbours they will invest below a level that is efficient.

This chapter contributes a new model of multi-layer networks. There are few existing papers where agents interact concurrently in multiple networks. König et al. (2014) model firms who compete in local markets after making cost-reducing investments in R&D networks. The price determining markets are modelled as overlapping coalitions. Chen et al. (2018) focus primarily on a single layer model, but they provide an extension where agents are connected in two networks with an action on each network, where strategic network effects are complements. Joshi et al. (2019) have a model where agents begin in a fixed network. They then form links to create a second network, where the benefits from network positioning are jointly derived from the two networks. Excluding our own model, we are not aware of any multi-layer models where inter-agent actions on both layers may be strategic substitutes.

This chapter proceeds as follows: Section 2 presents the model. Section 3 provides analysis of equilibrium, welfare properties, comparative statics, and stable equilibrium. Section 4 includes further discussion of the model and its implications. Section 5 discusses potential extensions. Section 6 concludes. All proofs are provided in the Appendix.

2 The Model

There are n agents, each existing in the set $N = \{1, \ldots, n\}$. These agents have the opportunity to invest effort in two non-excludable, local public goods, good 1 and good 2. Each public good has a distinct set of links that describe pairs of agents who share the benefits of public good investment. The set of links for good p is g^p, $\forall\, p \in \{1, 2\}$, which contains a binary element g^p_{ij} for each pair of agents $i, j \in N$. If a link exists in good p between i and j then $g^p_{ij} = 1$, otherwise $g^p_{ij} = 0$. Each set of links will be referred to as a separate *layer* of the network. Each layer is undirected, meaning that a link between agents i and j in layer p is a link between j and i, and $g^p_{ij} = g^p_{ji}\, \forall\, i, j \in N$, $\forall\, p \in \{1, 2\}$. As well, we assume that an agent does not link to herself, implying $g^p_{ii} = 0\, \forall\, i \in N$, $\forall\, p \in \{1, 2\}$. In $g + g^p_{ij}$, $g^p_{ij} = 1$ and all other links are as in g. Similarly, in $g - g^p_{ij}$, $g^p_{ij} = 0$ and all other links are as in g. The matrix whose i, j^{th} element is g^p_{ij} will be denoted $\boldsymbol{G}^p$.

In the layer g^p, any agent sharing a link with agent i is *connected* to i in g^p, and the set of all agents connected to agent i are i's *neighbours* in g^p, denoted $N_i(g^p) = \{j \in N \mid g^p_{ij} = 1\}$. i's *neighbourhood* in g^p is the union of i's neighbours and i. i's *cardinality* in g_p is the the number of neighbours that agent i has in g^p, denoted $\eta^p_i = \left|N_i(g^p)\right|$. An agent with no neighbours in g^p is considered to be in *autarky.*

Agents choose to invest in neither, one, or both of the public goods. An agent i's investment is $s_i = (s^1_i, s^2_i) \in S$, where S is a convex subset of $\mathbb{R}^2_+$ that includes the investment $(0, 0)$. The profile of all investments in the network is the two-dimensional vector $s = (s_1, \ldots, s_n) \in S^n$.

In each layer, the network structure and action set are consistent with many existing network games. While there are a small number of models addressed in Section 1 that include aspects of two networks, our model is the only model we are aware of that combines two actions on two separate layers into a multi-layer network with a single payoff function.

The payoff of any agent is defined by the following function:

$$\Pi_i(s \mid g) = f\left(s_i^1 + \sum_{j \in N_i(g^1)} s_j^1\right) - cs_i^1 + f\left(s_i^2 + \sum_{j \in N_i(g^2)} s_j^2\right) - cs_i^2 - \beta s_i^1 s_i^2. \quad (1)$$

The benefit function f is twice-differentiable and strictly concave, with $f(0) = 0$, $f'(\cdot) > 0$, $f''(\cdot) < 0$, and $f'(0) > c$. Because f is the same in both layers, comparative analysis is restricted to differences in the linking structure between the two layers. However, extending the model to allow for different benefit functions is straightforward and many of the conclusions persist. $c > 0$ is a fixed cost of investment that is constant across both layers.

The term $\beta s_i^1 s_i^2$ incorporates the cost of investing in two layers simultaneously. The marginal cost of investing in one layer for any agent i increases with their investment in the other layer. In a research context, this may represent an increased cost to a firm of spreading their efforts across multiple technologies. In keeping with this example, we will refer to β as a measure of *distraction*. As β increases, the cost of spreading effort across both layers increases as well. We assume that $\beta \geq 0$.

This cost term has some convenient properties. First, when $\beta = 0$, cost is additively separable into $cs_i^1 + cs_i^2$, and decisions in one layer are independent from actions in the other. Second, when $s_i^q = 0$, cost in layer g^p, cs_i^p, is independent of β—when an agent only invests in one of the two layers, their investment decision is independent of β.

Because an agent's investment provides benefit to all of his neighbours, this is a game of positive externalities. As well, if agent i makes an investment in layer g^p, and i and j are neighbours in g^p, then j's marginal benefit from investment in g^p will fall. Thus, for neighbours in layer g^p, investments in g^p are strategic substitutes. Because an agent's decision to invest in one layer increases the marginal cost for that agent of investment in the other, an agent's two investment opportunities are strategic substitutes for one another. To distinguish between these two different strategic effects, we refer to the investments of two agents connected in a layer as *inter-agent* strategic substitutes, whereas a single agent's two investments are *intra-agent* strategic substitutes.

The degree of substitution between an agent's two investments increases with β. When $\beta = 0$, the two layers are *disjoint*, equilibrium decisions in one layer are independent of equilibrium decisions in the other layer. As $\beta \to \infty$, agents will be unable to invest in both layers, and each agent must choose at most one layer to invest in. On intermediate values of β, agents may select investment in both layers, but the additional costs from the substitution effect of β may make them less likely to do so.

3 Analysis

Our model gives rise to four main questions: First, does equilibrium exist, and can we characterise the behaviour of all agents in equilibrium? Second, what are the welfare properties in equilibrium, measured both by aggregate payoff and the distribution of payoffs in the population? Third, how do equilibrium and welfare change with the model's parameters, specifically distraction and the linking structure? Finally, under what conditions do stable equilibria exist, and what are the welfare properties of stable equilibria?

As we have highlighted, when $\beta = 0$, an agent's two investment choices are independent, and the problem of maximising payoff for any agent is separable into maximising payoff on each layer. Thus, by setting $\beta = 0$, our model nests a base case presented by Bramoullé and Kranton (2007). We will provide comparison of our new results to this base case, but will not repeat their results in this chapter.

3.1 Equilibrium

The equilibrium concept used is Nash equilibrium. A strategy profile s^* is a Nash equilibrium if, for any agent i, strategy s_i^* is a strategy that maximises i's payoff, given all other agents invest according to s^*. More formally, s^* is a Nash equilibrium if

$$\Pi_i(s_i^*, s_{-i}^* \mid g) \geq \Pi_i(s_i, s_{-i}^* \mid g), \ \forall \, s_i \in S, \ \forall \, i \in N. \tag{2}$$

where $s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$ denotes the profile of investments by all agents excluding i.

We divide agents in equilibrium into three types: an agent who invests in both layers is a *dual-actor*, an agent who invests in only one layer is a *single-actor*, and an agent who does not invest at all is a *free-rider*. As well, an investor may be classified as a *specialist* if they are providing all of the investment in their neighbourhood, or as an *intermediate investor* if they are investing along with one or more neighbours. An equilibrium is *specialised* if all investors are specialists, *distributed* if there are no specialists, and a *hybrid* if it does not conform to either preceding category.

While S may be unbounded, for analysis we need only consider the feasible action space, $\tilde{S}$, defined to be the set of all $s_i \in S$ that may be optimal for an agent i—meaning that for some network (N, g) and set of actions s_{-i}, s_i may maximise i's payoff. The concavity of f ensures that $\tilde{S}$ is a compact subset of S. On this set, we make the following assumption.

Assumption 1. $f''(s_i^1)f''(s_i^2) > \beta^2 \; \forall \, s_i \in \tilde{S}$

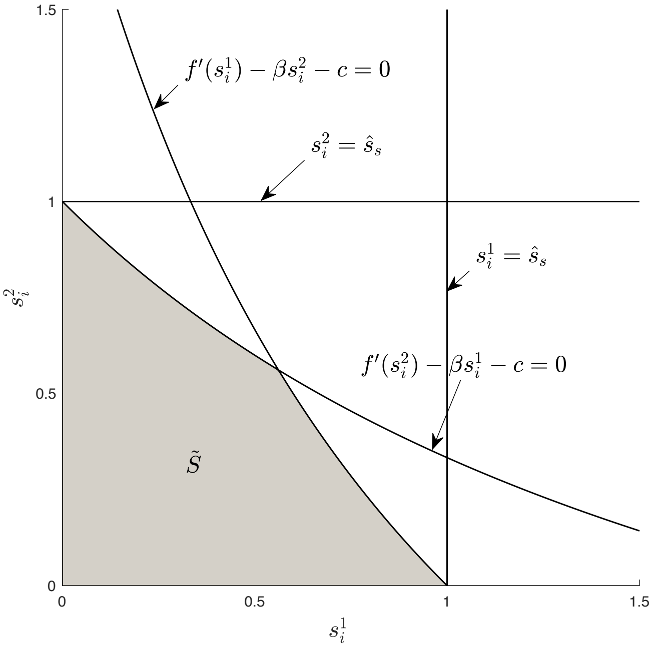
Assumption 1 is sufficient to guarantee that the payoff function $\Pi_i(s \mid g)$ is always concave on $\tilde{S}$. This, in turn, is used to show that any agent's optimal action on any network is uniquely determined by the actions of all other agents. We begin with the following theorem.

Theorem 1 (Existence). *Assume that Assumption 1 holds. On any network* (N, g) *there exists a specialist Nash equilibrium.*

The importance of specialist equilibria is reinforced in Section 3.4, where we discuss stable equilibria—equilibria for which a series of myopic best responses to a sufficiently small perturbation will converge on the original equilibria. There, we will show that stable equilibria must be specialist equilibria.

The proof of Theorem 1 relies on a complete characterisation of how all agents must act in equilibrium, which follows in Section 3.1.2. We prove the existence of equilibrium on any network with $n = 2$ then proceed inductively. For any network (N, g), we select an arbitrary agent k, and assume the existence of a specialised equilibrium on the reduced network $(N \setminus k, g)$. First, we determine when k's best-response action to the actions of the other agents does not force any of the other

Figure 1: The set $\tilde{S}$, for $f(x) = 2\log(x+1)$, $c = 1$, and $\beta = \frac{1}{2}$.



agents to change their action. Next, where this is not the case, we construct a finite sequence of action changes that must terminate in a specialised equilibrium.

3.1.1 The feasible set: $\tilde{S}$

An agent i acting in a single-layer network will have a profitable investment opportunity if their *local investment*, the sum of all of their neighbour's investments, is less than $\hat{s}_{\mathrm{s}}$, where $\hat{s}_{\mathrm{s}}$ is the unique solution to $f'(\hat{s}_{\mathrm{s}}) = c$ (see Bramoullé and Kranton 2007). i's investment is $s_i = \hat{s}_{\mathrm{s}} - \bar{s}_i$, with $\bar{s}_i = \sum_{j \in N_i(g)} s_j$. We extend this to a two-layer case by letting $\bar{s}_i^p = \sum_{j \in N_i(g^p)} s_j^p \; \forall \, p \in \{1, 2\}$, and conclude that an agent i will never invest in either layer if $\bar{s}_i^p \geq \hat{s}_{\mathrm{s}} \; \forall \, p \in \{1, 2\}$.

When $\bar{s}_i^p < \hat{s}_{\mathrm{s}}$ and $\bar{s}_i^q \geq \hat{s}_{\mathrm{s}}$, then i's actions are relatively straightforward. i will invest only in layer g^p, making investment $s_i = \{\hat{s}_{\mathrm{s}} - \bar{s}_i^p, 0\}$.

When both $\bar{s}_i^p < \hat{s}_{\mathrm{s}}$ and $\bar{s}_i^q < \hat{s}_{\mathrm{s}}$, for $p \neq q$, then i's decision is more complex. The marginal benefit of an investment of s_i^p is $\frac{\partial \Pi_i(s|g)}{\partial s_i^p} = f'(s_i^p + \bar{s}_i^p) - c - \beta s_i^q$. Therefore, for any agent i in equilibrium, it must always be the case that

$$f'(s_i^1 + \bar{s}_i^1) - \beta s_i^2 - c \leq 0 \text{ and} \tag{3}$$

$$f'(s_i^2 + \bar{s}_i^2) - \beta s_i^1 - c \leq 0. \tag{4}$$

Otherwise, agent i would increase investment in any layer where marginal payoff is positive. Further, we may assume that Equations (3) and (4) hold with strict equality whenever i makes positive investments in layers g^1 or g^2, respectively, and thus both equations must hold with strict equality in the case of an interior solution—when i is a dual-actor. Assumption 1 guarantees that the boundaries of inequalities (3) and (4) will intersect at most once. The feasible set, $\tilde{S}$, is the set constrained by inequalities (3) and (4) along with $s_i^1 \leq \hat{s}_{\mathrm{s}}$ and $s_i^2 \leq \hat{s}_{\mathrm{s}}$, as shown in Figure 1.

3.1.2 Characterisation of equilibrium

To fully describe the equilibrium actions of a single agent, we take an arbitrary agent i in a network (N, g) and let $p \in \{1, 2\}$ be the layer in which local investment for agent i is weakly lesser, and q the other layer. We have the following proposition.

Proposition 1. *Let i be an agent in the network (N, g), and let $\bar{s}_i^p \leq \bar{s}_i^q$ for $p, q \in \{1, 2\}$, $p \neq q$. Assume that Assumption 1 holds. The following four conditions must all be met in equilibrium:*

1. *If $\bar{s}_i^p \geq \hat{s}_{\mathrm{s}}$, then i will be a free-rider and make investment $s_i = (0, 0)$.*

2. *If* $\bar{s}_i^p < \hat{s}_\text{s}$ *and* $\hat{s}_\text{s} - \bar{s}_i^p \geq \frac{f'(\bar{s}_i^q)-c}{\beta}$, *then* i *will be a single-actor and make investments* $s_i^p = \hat{s}_\text{s} - \bar{s}_i^p$ *and* $s_i^q = 0$.

3. *If* $\bar{s}_i^p = \bar{s}_i^q < \hat{s}_\text{s}$, *then* i *will make investment* $s_i = (\tilde{s}_i, \tilde{s}_i)$, *where* $f'(\tilde{s}_i + \bar{s}_i^1) - \beta \tilde{s}_i - c = 0$.

4. *Otherwise,* i*'s unique optimal investment must involve investment in both layers.*

Using Proposition 1 we may classify how any agent must act in a specialised equilibrium. Any free-rider will invest $(0, 0)$, and must have local investment weakly greater than $\hat{s}_\text{s}$ in each layer. A single-actor will invest $\hat{s}_\text{s}$ in the layer in which no neighbours are investing, and must be connected to at least one investing agent in the other layer. Finally, if an agent is connected to no investors in either layer then they must be a dual-actor, making investment $(\hat{s}_\text{d}, \hat{s}_\text{d})$, where $f'(\hat{s}_\text{d}) - \beta \hat{s}_\text{d} - c = 0$.

3.2 Welfare properties

The first measure of welfare we will use is the aggregate payoff of all agents, which is defined as

$$W(s \mid g) = \sum_{i \in N} \left[f\left(s_i^1 + \bar{s}_i^1\right) - c s_i^1 + f\left(s_i^2 + \bar{s}_i^2\right) - c s_i^2 - \beta s_i^1 s_i^2 \right] \tag{5}$$

An equilibrium profile s is *efficient* if there is no other action profile that strictly increases welfare. That is, there is no $s' \in S^n$ such that $W(s' \mid g) > W(s \mid g)$. As well, we will analyse the distribution of payoffs within a population, under the assumption that a narrower distribution is more equitable. A key measure we will use is the minimal payoff to any agent in equilibrium.

First, we consider aggregate payoffs in a network, and determine how equilibrium decisions relate to efficiency. There are two elements of an individual agent's self-interested decision making that may create a divergence from efficient outcomes. The first, where an agent underinvests relative to an efficient level in each layer, parallels the discussion of disjoint layers. The second relates to an agent's layer choice, and how this affects other agents.

Within each layer, all agents in an efficient profile who are making a positive investment must invest such that $\frac{\partial W(s|g)}{\partial s_i^p} = 0$, which implies that

$$f'\left(s_i^p + \bar{s}_i^p\right) - \beta s_i^q + \sum_{j \in N_i(g^p)} f'\left(s_j^p + \bar{s}_j^p\right) - c = 0. \tag{6}$$

where g^p is the layer in which i is investing and g^q is the other layer. However, in equilibrium, $f'\left(s_i^p + \bar{s}_i^p\right) - \beta s_i^q - c = 0$ for any i investing in layer g^p, and because $f'(\cdot) > 0$, the term $\sum_{j \in N_i(g^p)} f'\left(s_j^p + \bar{s}_j^p\right)$ must be strictly positive. This guarantees that any agent who invests in equilibrium will always underinvest relative to an efficient level.

In any layer of a specialised equilibrium, only non-investors may have links to investing agents, meaning the payoffs for single-actors and dual-actors are fixed. From each layer, a dual-actor will receive payoff $f(\hat{s}_\text{d}) - c\hat{s}_\text{d} - \frac{1}{2}\beta\hat{s}_\text{d}^2$. A single-actor receives $f(\hat{s}_\text{s}) - c\hat{s}_\text{s}$ from the layer in which they are investing, and at least $f(\hat{s}_\text{d})$ from the other layer. A free-rider must have local investment of at least $\hat{s}_\text{s}$ in each layer, otherwise they would invest themselves, which ensures that the payoff that a free-rider receives from each layer is at least $f(\hat{s}_\text{s})$. This leads to the following proposition.

Proposition 2. *In any specialist equilibrium on the network* (N, g)*, all dual-actors will receive payoff less than that of any other agent.*

In Section 3.3 we examine the parameter values for which dual-actors may exist in equilibrium, concluding that parameterisations that exclude dual-actors will in turn prevent the most unequal equilibria from occurring.

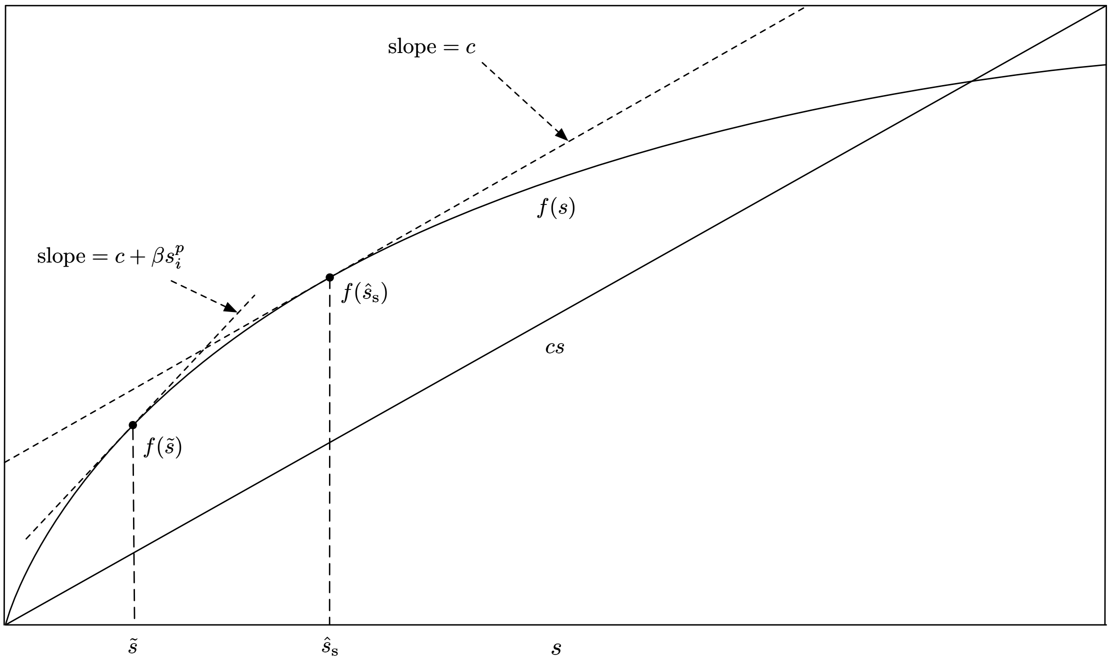
3.3 Comparative statics

We compare the strategic implications and welfare effects of changing two variables, β and g. This is measured according to *second-best* equilibrium profiles; an equilibrium s^* is second-best if and only if there is no other equilibrium $s^{*\prime}$ such that $W(s^{*\prime} \mid g) > W(s^* \mid g)$.

As β increases, it has multiple effects on an agent's ability to profitably invest in both layers concurrently. Directly, β affects the benefit from an agent's investments, so when β rises an agent investing in both goods will see their absolute and marginal costs increase. Holding actions constant, any agent investing in both goods will have a strictly lower payoff.

A secondary effect of an increase in β is that it expands the opportunity for agents contributing in one layer to avoid contribution in the other layer. For an agent i who makes an investment in layer g^p to not invest in g^q, his local investment in g^q, $\bar{s}_i^q$, must be sufficiently high that marginal return from any new investment will not exceed marginal costs, as is set out in Lemma 3. When β increases, these marginal costs will increase, and the threshold level of local investment required to sustain a non-investment for i in g^q falls, expanding i's ability to free-ride in

Figure 2: Benefit from investment



that layer.

Figure 2 illustrates this effect. We assume that for some agent i, $\bar{s}_i^p < \bar{s}_i^q$, thus if i's equilibrium investment is in a single layer i's investment will be $s_i^p = \hat{s}_s - \bar{s}_i^p$ in layer g^p. Then the optimal local investment for i in layer g^q is $\tilde{s}$, where $f'(\tilde{s}) = c + \beta s_i^p$. For any level of local investment above $\tilde{s}$, i will not invest in layer g^q in equilibrium, whereas if local investment is below $\tilde{s}$ then i must be a dual-actor, as set out in Proposition 1, statement 2. As β increases, $\tilde{s}$ decreases, which increases the range of local investment in layer g^q for which i's equilibrium action is to invest in a single layer.

If s^* is a second-best equilibrium profile on the network (N, g), then the addition of the link g_{ij}^p may have three effects. If both $s_i^{p*} = 0$ and $s_j^{p*} = 0$ then $W(s^* \mid g + g_{ij}^p) = W(s^* \mid g)$. s^* is still an equilibrium profile which yields the same welfare, and there may be another equilibrium profile where either or both of $s_i^p > 0$ and $s_j^p > 0$ which yields higher welfare. If either $s_i^{p*} > 0$ or $s_j^{p*} > 0$, then $W(s^* \mid g + g_{ij}^p) > W(s^* \mid g)$. s^* remains an equilibrium profile, and the new link passes additional benefit to a new node, thus second-best equilibrium in the new network must be strictly higher.

The final effect is where partial substitutes differ most from the case of disjoint layers. Suppose that both $s_i^{p*} > 0$ and $s_j^{p*} > 0$. Then, s^* is no longer an equilibrium in the network $(N, g + g_{ij}^p)$, and second-best welfare may increase or decrease. Holding initial investments constant, after i and j are linked in layer g^p, benefits will increase for both agents while costs will remain constant. However, because marginal benefit in layer g^p will decrease for both agents, it will no longer be an equilibrium and at least one of the agents will decrease their investment. Knock on effects will be multiple, supposing agent j reduces s_j^p, j's marginal cost in layer g^q, $c + \beta s_j^p$, will fall, and j may also increase s_{jq} in equilibria. As well, the initial decreases in s_j^p will result in the marginal benefit increasing for all $k \in N_j(g^p)$, and these agents may then increase their investment in g^p. As these actions may effect aggregate payoff both positively or negatively, the effect of the new link g_{ij}^p is indeterminate.

The following examples shows both consequences of an additional link.

Example 1 (Negative Effect)**.** *Consider the three-agent networks in Figure 3. We continue to use the model with $f(x) = 2\log(x+1)$, $\beta = \frac{1}{2}$, and $c = 1$. The initial network is shown in Figure 3a, along with each agent's investment in the second-best equilibrium, for which aggregate payoff is approximately* 5.318*. Note that in g^1 all investment borne by the most central agent,* 1*, leading to the second-best*

Figure 3: New link with a negative effect

(a) Before (b) After

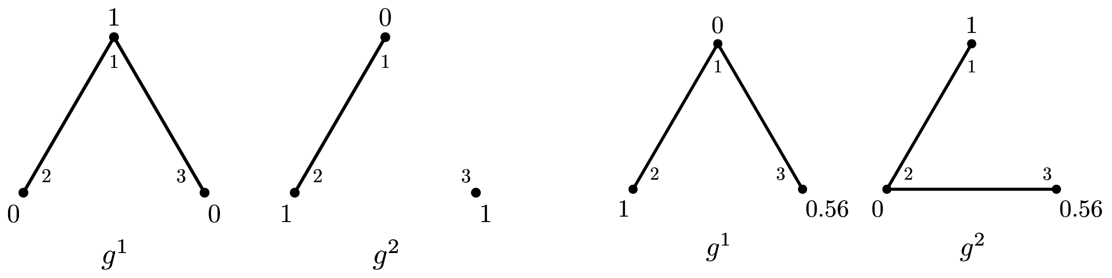
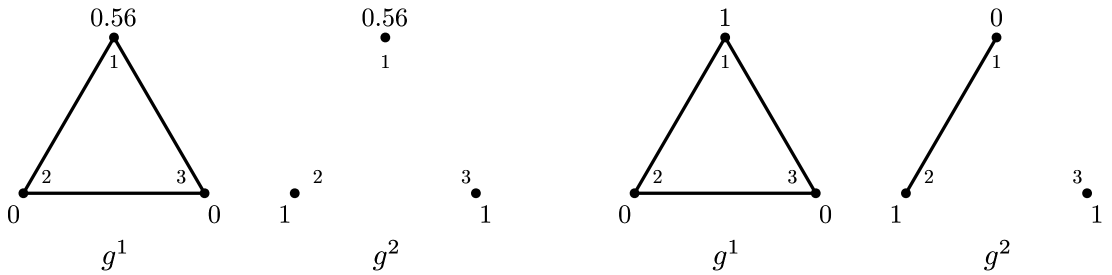


Figure 4: New link with a positive effect

(a) Before (b) After



equilibrium for that layer independently.

In Figure 3b, a link has been added between agents 2 *and* 3 *in layer* g^2*. While both agents had been investing before in Figure 3a, due to the new link they must in aggregate invest less in* g^2*. This, in turn, prevents agent* 1 *from free-riding in* g^2*, and* 1 *can't bear all of the investment in* g^1*. In the second-best equilibrium, which is shown in Figure 3b, the new link in layer* g^2 *results in a considerably worse aggregate outcome in layer* g^1*, and the total aggregate payoff is approximately* 5.037*. Aggregate payoff in the second-best equilibrium has fallen by about* 0.2808*.*

Example 2 (Positive Effect)**.** *Consider the three-agent networks in Figure 4, and the model with* $f(x) = 2\log(x+1)$*,* $\beta = \frac{1}{2}$*, and* $c = 1$*. The initial network is shown* *in Figure 4a, along with each agent's investment in the second-best equilibrium, which has aggregate payoff of approximately* 3.057*. Because there are no links in layer* g^2*, all agents must invest in this layer in any equilibrium, and they are less able to invest in layer* g^1 *where any investment is shared.*

In Figure 4, a link has been added between agents 1 *and* 2 *in layer* g^2*. Then, agents* 1 *and* 2 *must reduce their aggregate investment in* g^2*, which will benefit both agents. In the second-best equilibrium shown, agent* 1 *can free-ride off of agent* 2*'s investment in* g^2*, freeing up agent* 1 *to invest a greater amount in layer* g^1 *to the benefit of all agents. Aggregate payoff is approximately* 5.318*, and the improvement in the aggregate payoff in second-best equilibrium is approximately* 2.261*.*

3.4 Stability of equilibrium

An equilibrium is *stable* if, after a sufficiently small perturbation of the equilibrium investment profile, a series of myopic best responses by each agent will converge back to equilibrium. Agent i's best response to the profile of all other agents' investments is defined

$$r_i(s_{-i} \mid g) = \arg\max_{s_i}\{\Pi_i(s_i, s_{-i} \mid g)\}. \tag{7}$$

The profile of all agents' best responses is determined by $r(s \mid g) : S^n \to S^n$. Define the series $r^t(s \mid g) = r(r^{t-1}(s \mid g) \mid g)$ with $r^0(s \mid g) = s$. Then, the equilibrium s^* is stable if there exists some $\rho > 0$ such that, for any $\boldsymbol{\epsilon} \in \mathbb{R}_+^{n\times 2}$ with $\left|\epsilon_i^p\right| < \rho$ and $s_i^{p*} + \epsilon_i^p \geq 0$, $\forall\, i \in N$, $\forall\, p \in \{1, 2\}$, $\lim_{t\to\infty} r^t(s^* + \boldsymbol{\epsilon}) = s^*$.

Let s^* be an equilibrium, and suppose i is an intermediate investor in layer g^p. Then if i's neighbour in g^p increases his investment, i's best response may either be to decrease his investment in g^1 or to change his layer choice. When a permutation of equilibrium is such that investments all weakly increase in one layer, in the first step of myopic best responses all investments in the same layer will be weakly lower, while all investments in the opposite layer will be weakly higher. In every step, this pattern will reverse, and this oscillating pattern is key in demonstrating that any equilibrium with intermediate investments may be permuted in such a manner that they never converge to the original equilibrium, ensuring a stable equilibrium must be specialised. This is essential in proving the following theorem.

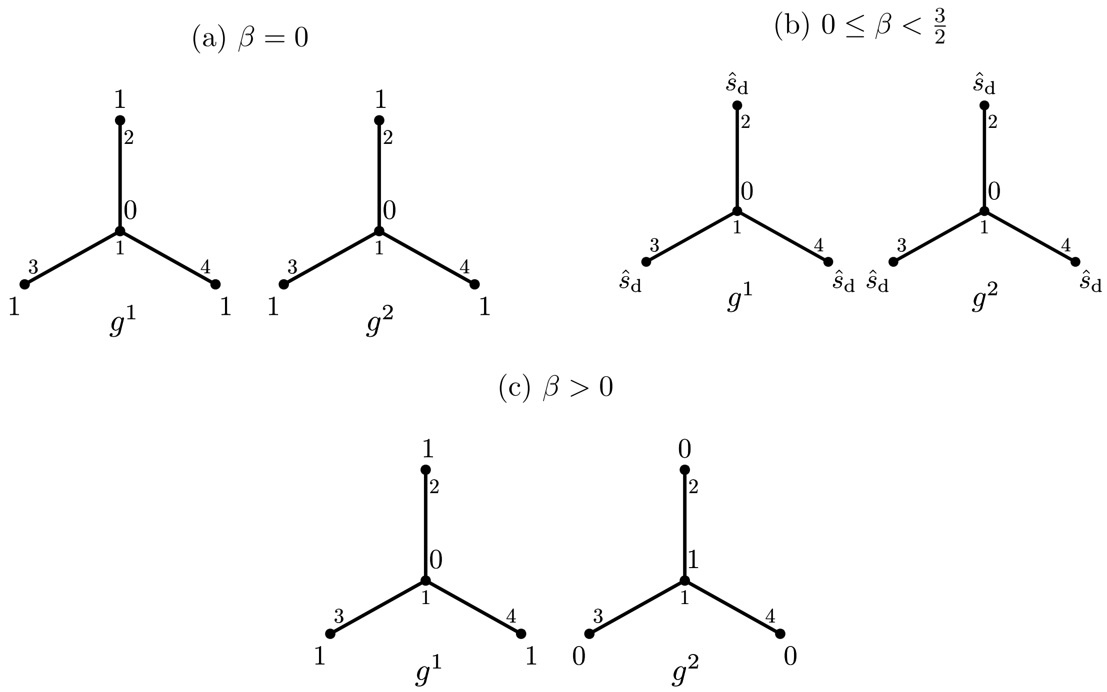
Theorem 2. *Assume Assumption 1 holds. An equilibrium is stable if and only if the set of agents N can be partitioned into four disjoint sets, L, I^1, I^2, and D, where*

1. *$\forall\, p \in \{1, 2\}$, $D \cup I^p$ is a maximal independent set in layer g^p, and*
2. *$\forall\, \ell \in L$ and $\forall\, p \in \{1, 2\}$, ℓ is linked in layer g^p to either*
   - *(a) more than $\frac{\hat{s}_{\mathrm{s}}}{\hat{s}_{\mathrm{d}}}$ agents in set D, or*
   - *(b) at least one agent in I^p and more than one agent in $D \cup I^p$,*

and the actions of all agents are as follows: $s_d = (\hat{s}_{\mathrm{d}}, \hat{s}_{\mathrm{d}})\,\forall\, d \in D$, $s_{i^1} = (\hat{s}_{\mathrm{s}}, 0)\,\forall\, i^1 \in I^1$, $s_{i^2} = (0, \hat{s}_{\mathrm{s}})\,\forall\, i^2 \in I^2$, and $s_\ell = (0, 0)\,\forall\, \ell \in L$.

As the value of β increases, $\hat{s}_{\mathrm{d}}$ decreases becauese $f'(\hat{s}_{\mathrm{d}}) - \beta\hat{s}_{\mathrm{d}} - c = 0$. With $\hat{s}_{\mathrm{s}}$ fixed, the ratio $\frac{\hat{s}_{\mathrm{s}}}{\hat{s}_{\mathrm{d}}}$ then increases as β increases. As a result, as the value of β rises, Theorem 2 condition 2a indicates that, if a free-rider is not free-riding from at least one single-actor, the number of dual-actors they must be connected to will

Figure 5: Stable Equilibria



rise. Ultimately, distraction may have two effects on stable equilibria, the payoff for dual-actors and their neighbours will fall as the level of distraction rises, when actions remain constant, but a higher level of distraction may preclude equilibria that feature agents free-riding off of dual-actors, which will increase equity and may increase aggregate payoff, as the following example illustrates.

Example 3. *Let* $n = 4$*,* $f(x) = 2\log(x+1)$ *and* $c = 1$*, so that*

$$\Pi_i(s \mid g) = 2\log\left(s_i^1 + \sum_{j\in N_i(g^1)} s_j^1 + 1\right) - s_i^1 + 2\log\left(s_i^2 + \sum_{j\in N_i(g^2)} s_j^2 + 1\right) - s_i^2 - \beta s_i^1 s_i^2, \quad (8)$$

and assume that the network is a star in each layer, with the same central agent *in both layers. Then* $\hat{s}_{\mathrm{s}} = 1$ *and* $\hat{s}_{\mathrm{d}} = \frac{\sqrt{1+6\beta+\beta^2}-(1+\beta)}{2\beta}$*, which is decreasing in* β*. When* $\beta = 0$*, the only stable equilibrium is depicted in Figure 5a. In this equilibrium, the central agent receives* $\Pi_1 \approx 5.54$*, the peripheral agents (agents 2–4) receive* $\Pi_{-1} \approx 0.77$*, and aggregate payoff is* $W \approx 7.86$*. Figure 5b depicts the equivalent stable equilibrium when* β *is less than* $\frac{3}{2}$*. Now* $\Pi_1 = 4\log(3\hat{s}_{\mathrm{d}} + 1)$ *and* $\Pi_{-1} = 4\log(\hat{s}_{\mathrm{d}} + 1) - 2\hat{s}_{\mathrm{d}}$*. As there is no change in strategy apart from reducing investment, distraction makes all agents worse off. However, for values of* β *above this range, equilibrium 5b cannot persist, as the central agent's local investment is insufficient to support free-riding. Then, the equilibrium in Figure 5c, which is stable for all* $\beta > 0$*, is the only stable equilibrium. In this equilibrium,* $\Pi_1 \approx 3.16$*,* $\Pi_{-1} \approx 1.77$*, and* $W \approx 8.48$*. Because distraction forces agents into a different set of investments versus when layers are disjoint, distraction increases aggregate payoff and the distribution of payoffs is more equitable.*

4 Discussion

In each layer, agents who contribute receive lower payoffs than those who do not, which is why dual-actors must be the least well off of all agents. Holding investments fixed, increasing distraction will penalise dual-actors further. However, we've shown that, as distraction increases, the minimal payoff for any agent in a network will eventually become higher when dual-action becomes unsustainable. In essence, being the lowliest agent becomes so unpalatable that these agents are forced to stop allowing their neighbours to profit at their expense in both layers, and where possible, other agents will take their place and become investors.

When distraction rises, dual-actors gain a comparative strategic advantage over their free-riding neighbours. Because a dual-actor is distracted, the marginal benefit they receive from investing is lower. When agents have different marginal benefits in one layer, the game becomes similar in character to Allouch's (2015) local public good game with heterogeneous wealth. There, wealthier agents have greater marginal benefit from public investment, and in each local neighbourhood the wealthiest agents invest to the benefit of their poorer neighbours. In our model, after fixing the investments of all agents in one layer, agents will have heterogeneous payoff functions from investment in the other layer.

Gagnon and Goyal's (2017) model provides similar lessons in comparative advantages within networks. Agents have a binary market action that provides a fixed payoff to all, and a binary network action with increasing benefit as the number of neighbours taking this action rises. In the case of strategic substitutes, taking the market action reduces the rate at which network benefits increase. In this case, highly connected agents who are connected primarily to highly connected agents, those who are benefitting most from the network action, may choose not to take the market action as it will reduce their network benefits. Less connected agents will have less to lose and will select the market action, and thus the market action serves to decrease inequality. In our model, the connected actions are both network actions, but benefitting from one action (or non-action) may still create disadvantages in the other network, leading to similar effects.

5 Extensions

Because strategic network effects are substitutes, actions tend to separate. As more agents invest in one layer, the marginal payoff to any agent investing in that layer weakly decreases. This, in effect, limits the amount of investment that any layer will receive. In contrast, some networks may feature complementary network effects. Then, as more agents invest in a layer, marginal payoff to agents investing in that layer will weakly increase, which may further draw agents to invest in that layer. We could see a pooling effect, as agents might coordinate and only invest in complementary layers when a sufficient number of their neighbours are doing so as well.

To allow for a more robust set of potential inter- and intra-agent interactions, we must find a simpler model. Chen et al. (2018) model socially connected criminals who engage in a second network activity, and we use their model as a starting point. Here, an agent's payoff is modelled according to the payoff function

$$\Pi_i(s \mid g) = \alpha^1 s_i^1 + \alpha^2 s_i^2 - \left[\frac{1}{2}\left(s_i^1\right)^2 + \frac{1}{2}\left(s_i^2\right)^2 + \beta s_i^1 s_i^2\right] + \delta \sum_{j \in N_i(g)} \left(s_i^1 s_j^1 + s_i^2 s_j^2\right). \quad (9)$$

The parameter $\beta \in (-1, 1)$ describes the nature of intra-agent strategic interactions, and the parameter $\delta > 0$ ensures that inter-agent strategic interactions are complements. We extend the model in such a manner that each action has its own layer of the network as follows:

$$\Pi_i(s \mid g) = \alpha^1 s_i^1 - \frac{1}{2}\left(s_i^1\right)^2 + \beta s_i^1 s_i^2 + \alpha^2 s_i^2 - \frac{1}{2}\left(s_i^2\right)^2 + \delta^1 \sum_{j \in N_i(g^1)} s_i^1 s_j^1 + \delta^2 \sum_{j \in N_i(g^2)} s_i^2 s_j^2. \quad (10)$$

Let $\Gamma(N, S, g)$ denote the game on network (N, g) with action set S^n and this payoff function. Note that $\frac{\partial^2 \Pi_i(s|g)}{\partial s_i^1 \partial s_i^2} = \beta$, and so $\beta \in (-1, 1)$ determines the nature of intra-agent strategic interactions. For $\beta > 0$ an agent's two actions are strategic complements, and $\beta < 0$ implies that an agent's two actions are strategic substitutes. Because $\frac{\partial^2 \Pi_i(s|g)}{\partial s_i^p \partial s_j^p} = \delta^p$, strategic interactions on the layer g^p are determined by the parameter $\delta^p \in (-1, 1)$, with $\delta^p > 0$ implying actions on that layer are strategic complements and $\delta^p < 0$ describing actions which are strategic substitutes. In an extension, Chen et al. (2018) describe how their model may be extended to a multi-layer framework and provide existence results. Our use of this model goes beyond these results in allowing for strategic substitutes on networks and allowing for the action set to be bounded, for instance, enforcing that all actions must be positive.

Bramoullé, Kranton and D'Amours (2014) demonstrate how this new model can incorporate their public good model in Bramoullé and Kranton (2007), and we apply a similar extension to the main model in this paper. Taking $\beta = 0$, then in each layer $\frac{\partial \Pi_i(s|g)}{\partial s_i^p} = \alpha^p - s_i^p + \delta^p \bar{s}_i^p$. Setting marginal payoff equal to zero, excluding where investment is bound to be positive, implies that $r_i(s \mid g) = \max\left\{0, \alpha^p - \delta^p \bar{s}_i^p\right\}$. Setting $\alpha^p = \hat{s}_{\mathrm{s}}$ and $\delta^p = 1$, this is identical to the best-reply function in our main model when $\beta = 0$. Since agents in both models have identical best-reply functions, they must have identical sets of Nash equilibria.

To ensure the tractability of this new model, we first establish that $\Gamma(N, S, g)$ is a potential game (see Monderer and Shapley 1996). In a potential game, there exists a potential function ϕ such that, for all $s_i, s_i' \in S$ and $s_{-i} \in S^{n-1}$,

$$\phi(s_i, s_{-i}) - \phi(s_i', s_{-i}) = \Pi_i(s_i, s_{-i} \mid g) - \Pi_i(s_i', s_{-i} \mid g). \tag{11}$$

For a potential game, the set of Nash equilibria is isomorphic to the set of maxima and saddle points of the potential function.

Redefining s to be the vector $(s_{11}, \dots, s_{n1}, s_{12}, \dots, s_{n2})^\mathsf{T}$ we propose the following.

Proposition 3. *The function*

$$\phi(s) = \begin{pmatrix} \alpha^1 \mathbf{1} \\ \alpha^2 \mathbf{1} \end{pmatrix}^\mathsf{T} s - \frac{1}{2} s^\mathsf{T} \left(\boldsymbol{I} - \begin{bmatrix} \delta^1 \boldsymbol{G}^1 & \beta \boldsymbol{I} \\ \beta \boldsymbol{I} & \delta^2 \boldsymbol{G}^2 \end{bmatrix} \right) s \tag{12}$$

is a potential function for the game $\Gamma(N, S, g)$.

By determining the parameters under which the potential function is uniquely concave, we can then prove the following theorem.

Theorem 3. *Let*

$$\upsilon^p = \begin{cases} \delta^p \lambda_{\max}(\boldsymbol{G}^p) & \text{if } \delta^p > 0 \\ \delta^p \lambda_{\min}(\boldsymbol{G}^p) & \text{if } \delta^p < 0. \end{cases} \tag{13}$$

If

$$(1 - \upsilon^1) > 0 \text{ and} \tag{14}$$

$$\beta^2 < (1 - \upsilon^1)(1 - \upsilon^2) \tag{15}$$

then the game $\Gamma(N, S, g)$ *has a unique equilibrium on the action space* S.

This result is consistent with Chen et al. (2018, Theorem 6), which shows that when $\delta^1 > 0$, $\delta^2 > 0$, and $S = \mathbb{R}^2$, then $\min\{(1 - \upsilon^1), (1 - \upsilon^2)\} > |\beta|$ implies that there is a unique equilibrium. Our result allows for a more general parameter space, the ability to restrict the action space, and expands the threshold for which a unique equilibrium must exist.

6 Conclusion

Individuals and firms face choices in how to allocate their resources across existing opportunities. If our neighbour is willing to contribute her own resources towards our shared benefit, we may invest our own resources elsewhere and exploit our neighbour's generosity. Such strategic incentive to exploit neighbours may lead to outcomes where some people contribute and others free-ride. We explore these incentives in the context of innovation, where firms have opportunity to invest in researching two different technologies, and research achievements are shared with neighbouring firms.

In our model a group of agents is connected by two distinct sets of links, with each set describing pairs of agents who share benefit from investment in two different local public goods. Marginal benefit is declining in local investment, so agents have incentive to reduce investment when neighbours' investments increase, and inter-agent investments are strategic substitutes. When an agent invests in both goods, the cost of each investment increases, and intra-agent investments are strategic substitutes. We have shown how an increase in an agent's costs can be beneficial; when agent i's return on investment is greater than that of his neighbour j, because j is investing in the other technology, it may ensure that in equilibrium i will invest and j will benefit.

Our model provides a framework to analyse how investments in one good affect investments in the other, and to understand the resulting distribution of payoffs across the population. From each good, non-investing agents always receive higher payoff than investors, and payoff is decreasing in the level of investment. However, because investment in one good reduces the profitability of investment in the other, combining two networks and two public goods may have a tendency to balance payoffs between the two goods and increase equity. As the cost of investing in both goods increases, the conditions under which a single agent may invest in both goods cease to exist, and as a result the minimal achievable payoff in equilibrium increases.

To conclude, we will acknowledge some of our model's limitations and remark on potential areas for extension. Agents and public goods are heterogeneous only in linking structure. A more robust model might include heterogeneous wealth; and, if wealthier agents have a higher propensity to invest in the public good, this could overwhelm the strategic effects of investment. While assigning the same payoff function to each network ensures that the results reflect differences in linking structure, if each network had a different payoff function we might determine how different strategic affects cause agents to act. Because inter-agent actions are strategic substitutes, an agent's neighbours' investments push that agent towards investment in the other network; but, if actions are complements we might see agents with incentive to pool investments together in one of the networks.

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Appendix

Proof of Theorem 1. Please note that this proof requires Proposition 1, which is proved later in this section.

This proof proceeds by induction. First we show that, when there are only 2 agents, in any network structure there is always a Nash equilibrium. For the inductive step, we select an arbitrary node k in any network and assume that there is at least one equilibrium in the network $g \setminus k$. Assigning the actions from this equilibrium to the agents in N given the network g, we determine k's best-response actions. When k's actions have no effect on k's neighbours, then this is an equilibrium. When k's actions do effect his neighbours, we show that a cascading sequence of best responses by all newly affected nodes must be finite, ultimately ending in an equilibrium.

Base Case: Assume $n = 2$.

Let $N = \{i, j\}$. Then, there are four possible network structures, and we may describe each possible set of links using the ordered pair $g = (g^1_{ij}, g^2_{ij})$. If $g = (0, 0)$, then the investments $s_i = (\hat{s}_\mathrm{d}, \hat{s}_\mathrm{d})$ and $s_j = (\hat{s}_\mathrm{d}, \hat{s}_\mathrm{d})$ are an equilibrium. If $g = (0, 1)$, then the investments $s_i = (\hat{s}_\mathrm{s}, 0)$ and $s_j = (\hat{s}_\mathrm{d}, \hat{s}_\mathrm{d})$ are an equilibrium. The case where $g = (1, 0)$ is symmetric to $g = (0, 1)$, and therefore the investments $s_i = (0, \hat{s}_\mathrm{s})$ and $s_j = (\hat{s}_\mathrm{d}, \hat{s}_\mathrm{d})$ are an equilibrium. Finally, if $g = (1, 1)$, then the investments $s_i = (\hat{s}_\mathrm{s}, 0)$ and $s_j = (0, \hat{s}_\mathrm{s})$ are an equilibrium.

We have demonstrated that a specialised equilibrium exists for all possible networks with $n = 2$.

Inductive Step: Assume that for any $n \in \mathbb{Z}, n > 2$, a specialised equilibrium exists $\forall\, |N| < n$. We must show that for any network g with n agents, a specialised equilibrium exists.

Begin with the network (N, g), with $|N| = n$, and select any arbitrary agent k. Construct the reduced network, $(N \setminus k, g)$, by removing k and any links connected to k. By our inductive assumption, there is a specialised equilibrium profile on the network $(N \setminus k, g)$. Let this profile be s^*.

Assign the actions in s^* to the agents in $N \setminus k$. Then, $N \setminus k$ may be partitioned into four disjoint sets. Let D denote the set of dual-actors, with $s_d = (\hat{s}_\mathrm{d}, \hat{s}_\mathrm{d}) \,\forall\, d \in D$. Let I^1 denote the set of single-actors investing in layer g^1, with $s_{i^1} = (\hat{s}_\mathrm{s}, 0) \,\forall\, i^1 \in I^1$. Let I^2 denote the set of single-actors investing in layer g^2, with $s_{i^2} = (0, \hat{s}_\mathrm{s}) \forall\, i^2 \in I^2$. Finally, let L denote the set free-riders, with $s_\ell = (0, 0) \,\forall\, \ell \in L$.

Next, we will consider all the potential sets of neighbours that k may have in (N, g), and the actions that k will take assuming all neighbours are taking actions s^*. There are four simple cases where k's actions do not require any other agents to deviate from s^*, which are described below.

Case 1: $\exists\, i^1 \in I^1$ such that $i^1 \in N_k(g^1)$ and $\exists\, i^2 \in I^2$ such that $i^2 \in N_k(g^2)$

When k is linked to at least one single-investor in each layer, k's best response is to make investment $s_k = (0, 0)$. This will not change the local investment for any agent in $N \setminus k$ in either layer, and the action set $s^* \cup s_k$ is a specialised equilibrium on the network (N, g).

Case 2: $\exists\, i \in I^1 \cup D$ such that $i \in N_k(g^1)$ and $\forall\, j \in N_k(g^2)$, $j \in L \cup I^1$

In this case, k is linked to at least one investor in layer g^1, and k is only linked to non-investors in g^2. k's best response is to make investment $s_k = (0, \hat{s}_{\mathrm{s}})$. When k does so, the local investment of all of k's neighbours in g^1 will remain unchanged, and their actions described by s^* will remain optimal when links are added to k. When k makes the investment s_k, the local investment of all of k's neighbours in g^2 will increase. Because all of k's neighbours in g^2 are non-investors in g^2 in s^*, an increase in their local investments will ensure that non-investing remains optimal. Thus, the action set $s^* \cup s_k$ is a specialised equilibrium on the network (N, g).

Case 3: $\exists\, i \in I^2 \cup D$ such that $i \in N_k(g^2)$ and $\forall\, j \in N_k(g^1)$, $j \in L \cup I^2$

This case is symmetric to Case 2 with layers g^1 and g^2 reversed, and follows accordingly.

Case 4: $\forall\, i \in N_k(g^1)$, $i \in L \cup I^2$ and $\forall\, j \in N_k(g^2)$, $j \in L \cup I^1$

Here, k is linked only with non-investors in each layer. k therefore has no local investment in either layer, and k's optimal investment is $s_k = (\hat{s}_\mathrm{d}, \hat{s}_\mathrm{d})$. This action will increase the local investment for all of k's neighbours in both layers. Because all of k's neighbours were non-investors in s^*, after increasing local investment non-investing will remain optimal. Therefore, the action set $s^* \cup s_k$ is a specialised equilibrium on the network (N, g).

If none of Cases 1–4 hold, then it must be true that: after k employs his best response to all of the agents in $N \setminus k$ employing strategy s^*, there is at least on agent in $N \setminus k$ who will deviate from s^* in response to k's action. This occurs when k is linked to one (or more) dual-actors in a layer, but no single actors, and k's local investment in this layer is insufficient to support non-investment. After an investment by k, these neighbouring dual-actors must respond by reducing their own investments. This scenario may concurrently occur in both layers, for instance if k is linked to one dual-actor in each layer. Without loss of generality, we will proceed by assuming that this occurs in layer g^1 and that when all agents in $N \setminus k$ employ strategy s^*, $\bar{s}_k^1 \leq \bar{s}_k^2$.

We assume that k makes the investment $s_k = (\hat{s}_\mathrm{s}, 0)$. Then, alter the strategies employed by all agents according to the following sequence:

1. Assign the strategy $s^* \cup s_k$ to all agents in N.
2. The only set of agents, excluding k, for whom their currently strategy is not optimal is the set of dual-actors who are linked to k in layer g^1. Assume that all of these agents change their action to $(0, \hat{s}_\mathrm{s})$, so that they are now single-actors in layer g^2. This will change the subsets of N in the following way, where the subscript 'old' denotes the subsets prior to this step: $I^2 = I^2_\mathrm{old} \cup (D_\mathrm{old} \cap N_k(g^1))$ and $D = D_\mathrm{old} \setminus (D_\mathrm{old} \cap N_k(g^1))$.

3. (a) There may be free-riders who are unconnected to any single-actors in g^1, who now have insufficient local investment to support free-riding in g^1 (because at least one dual-actor to whom they are linked in g^1 became a single-actor in layer g^2 in step 2). If any such free-riders exist, take the set L' to be a maximal independent set of these free-riders in g^1. Then, assign the action $s_{\ell'} = (\hat{s}_{\mathrm{s}}, 0)$ to all agents in L', which results in the following two new sets: $I^1 = I^1_{\mathrm{old}} \cup L'$ and $L = L_{\mathrm{old}} \setminus L'$.

   (b) If L' is non-empty in step 3a, meaning there are agents switching from the set L to the set I^1, then this may cause further agents to need to change their action. Such agents could only be dual-actors who were linked to agents in L' in layer g^1, who now find themselves linked to single-actors. If any such-dual actors exist, let them compose the set D', and assign them the action $s_{d'} = (0, \hat{s}_{\mathrm{s}})$, so that they are now members of I^2. The change in sets is $I^2 = I^2_{\mathrm{old}} \cup D'$ and $D = D_{\mathrm{old}} \setminus D'$.

   (c) If a non-empty set of dual-actors become single-actors in g^2 in step 3b, then this may force additional free-riders who are connected to these agents in layer g^1 to begin investing. This mirrors the change that occurs in step 3a. If we continue to repeat steps 3a and 3b, then we will alternate between moving agents from L to I^1 and moving agents from D to I^2. Because L is finite, this sequence of repeated action changes must eventually terminate.

4. After step 2 and repeated applications of steps 3a and 3b the change

may be summarised: a subset of L have switched to I^1 and a subset of D have switched to I^2. The new members of I^1 will have no further effects on other members of the network; they may not have links to other agents in I^1 or else they would not have switched in the first place. When agents move from D to I^2, however, they may affect additional agents. This would only be previous members of I^2 who are now connected only to non-investors in g^1. If there are no such agents, the current action profile is a specialised equilbrium. Otherwise, let $I^{2'}$ be a maximal independent set of these agents in g^1, and assign all members of $I^{2'}$ the new action $s_{i^{2'}} = (\hat{s}_\mathrm{d}, \hat{s}_\mathrm{d})$. We now have the new sets $D = D_\mathrm{old} \cup I^{2'}$ and $I^2 = I^2_\mathrm{old} \setminus I^{2'}$.

5. (a) The change in step 4 will only affect one new set of agents: some free-riders may no longer have sufficient local investment in g^2. If there are no such free-riders the proof is complete, otherwise let L' be a maximal independent subset of these free-riders in g^2. Change the action for all of the agents in L to make them single-actors in g^2. The new subsets are $I^2 = I^2_\mathrm{old} \cup L'$ and $L = L_\mathrm{old} \setminus L'$.
   (b) As in step 3b, step 5a may require dual-actors to switch to single-action in layer g^1.
   (c) As in step 3c, steps 5a and 5b must terminate after a finite sequence of repetitions. At this point, a subset of agents in L will have switched to I^2, and a subset of agents in D will have switched to I^1.
6. After steps 5a–5c, we have a symmetric scenario to that preceding

step 4. If there are any members of I^1 who are no longer linked to any investors in g^2, we move them to D by changing their action to dual-action. If there are no such agents the current action profile must be a specialised Nash equilibrium.

7. After step 6, the scenario is symmetric to that preceding steps 5a–5c. We have either reached a symmetric Nash equilibrium, or a subset of agents will have to be moved from set L to set I^1 and a subset of agents may have to be moved from set D to set I^2.

8. At this point, step 4 repeats. Either we have a specialised equilibrium, or there are members of set I^2 who must be moved to set D.

If a repeated loop of steps 4–7 ever terminates, it must do so in a specialised Nash equilibrium. Now, note that steps 5a and 7 require that agents be moved from set L into another set, otherwise the algorithm will terminate. Because agents are never moved into set L, and set L must be finite to begin with, this algorithm must eventually terminate.

Therefore, a specialised equilibrium must exist with $n + 1$ agents, and the inductive step is complete.

$\square$

Proof of Proposition 1. Proposition 1 summarises a series of lemmas that govern how an individual agent in an equilibrium must act. The first ensures that any agent has a unique optimal investment, given the investment decisions of all other agents.

Lemma 1. *Given the investment decisions of all other agents, s_{-i}, an agent i has a unique profit maximising investment s_i.*

Proof. The Hessian matrix of $\Pi_i(s \mid g)$ at s_i is

$$\boldsymbol{H} = \begin{bmatrix} f''(s_i^1) & -\beta \\ -\beta & f''(s_i^2) \end{bmatrix} \tag{16}$$

By assumption, $f''(x) < 0 \,\forall\, x \in \mathbb{R}_+$, so the first leading principal minor of $\boldsymbol{H}$ is always negative. The second leading principal minor of $\boldsymbol{H}$ is $f''(s_i^1)f''(s_i^2) - \beta^2$. Assumption 1 tells us that $f''(s_i^1)f''(s_i^2) > \beta^2 \,\forall\, s_i \in \tilde{S}$, which implies that the second leading principal minor is positive $\forall\, s_i \in \tilde{S}$. Because all odd leading principal minors are always negative on $\tilde{S}$, and all even leading principal minors are always positive on $\tilde{S}$. then $\boldsymbol{H}$ is negative definite $\forall\, s_i \in \tilde{S}$, which in turn implies that $\Pi_i(s \mid g)$ is concave on $\tilde{S}$. By construction, $\tilde{S}$ is compact. A concave function on a compact set must have a unique maximum.

$\square$

Next we consider an agent's actions when it is optimal for an agent to invest in a single layer.

Lemma 2. *In Nash equilibrium, if an agent is a single-actor, then they will always invest in a layer g^p where*

$$p = \arg\min_x \bar{s}_{ix}. \tag{17}$$

Proof. Assume that agent i must be a single-actor, and that i is choosing between investment in layer g^1 or layer g^2. Assume that $\bar{s}_i^p \leq \bar{s}_i^q$. As well, since investment will not be profitable in g^q when $\bar{s}_i^q \geq \hat{s}_{\mathrm{s}}$, assume also that $\bar{s}_i^q < \hat{s}_{\mathrm{s}}$.

The return that i generates in either layer g^m in excess of making no investment is

$$f(\hat{s}_\mathrm{s}) - f(\bar{s}_i^m) - c(\hat{s}_\mathrm{s} - \bar{s}_i^m). \tag{18}$$

Then, the difference between an investment in g^p and an investment in g^q is

$$f(\hat{s}_\mathrm{s}) - f\left(\bar{s}_i^p\right) - c\left(\hat{s}_\mathrm{s} - \bar{s}_i^p\right) - \left[f(\hat{s}_\mathrm{s}) - f\left(\bar{s}_i^q\right) - c\left(\hat{s}_\mathrm{s} - \bar{s}_i^q\right)\right] \tag{19}$$

$$= f\left(\bar{s}_i^q\right) - f\left(\bar{s}_i^p\right) - c\left(\bar{s}_i^q - \bar{s}_i^p\right) \tag{20}$$

Because $f''(\cdot) < 0$ and $f'(\hat{s}_\mathrm{s}) = c$, $f'(x) > c \;\forall\, x \in [0, \hat{s}_\mathrm{s}]$. Then, given that $\bar{s}_i^p \leq \bar{s}_i^q < \hat{s}_\mathrm{s}$, it must also follow that $f\left(\bar{s}_i^q\right) - f\left(\bar{s}_i^p\right) > c\left(\bar{s}_i^q - \bar{s}_i^p\right)$ when $\bar{s}_i^p \neq \bar{s}_i^q$, which would establish that difference between investing in g^p and g^q must be positive. A utility maximising single-actor will, therefore, always invest in a layer with minimal local effort.

□

We may determine precisely when investment in a single layer will be optimal for any agent, and also what investment must be in this scenario.

Lemma 3. *Let s^* be an equilibrium in the network (N, g). Assume that there exists an agent i for whom $\bar{s}_i^p < \bar{s}_i^q$ and $\bar{s}_i^p < \hat{s}_\mathrm{s}$. Then i is a single-actor making investment $s_i^{p*} = \hat{s}_\mathrm{s} - s_i^p$ in layer g^p if and only if*

$$s_i^{p*} \geq \frac{f'(\bar{s}_i^q) - c}{\beta}. \tag{21}$$

Proof. Let $\bar{s}_i^p \leq \bar{s}_i^q$, and assume that agent i is investing in only one layer. By Lemma 2, we know that this must be layer g^p. Then i's investment is $s_i = \{s_i^p, 0\}$.

Suppose i is making investment $s_i^p \neq \hat{s}_{\mathrm{s}} - \bar{s}_i^p$ in layer g^p. Then i's marginal payoff in layer g^p, is

$$f'(s_i^p + \bar{s}_i^p) - c \neq f'(\hat{s}_{\mathrm{s}} - \bar{s}_i^p + \bar{s}_i^p) - c \tag{22}$$

$$= f'(\hat{s}_{\mathrm{s}}) - c \tag{23}$$

$$= 0 \tag{24}$$

Because i's marginal payoff in layer g^p is not zero, i may improve his payoff by changing his investment, and i cannot be making an equilibrium investment. Thus, we may conclude that $s_i^p = \hat{s}_{\mathrm{s}} - \bar{s}_i^p$.

Now, given $s_i^p = \hat{s}_{\mathrm{s}} - \bar{s}_i^p$ and $s_i^q = 0$, i's marginal payoff from investment in layer g^q is $f'(\bar{s}_i^q) - \beta(\hat{s}_{\mathrm{s}} - \bar{s}_i^p) - c$. i may only invest zero in layer g^q if i's marginal payoff is weakly negative, that is

$$f'(\bar{s}_i^q) - \beta(\hat{s}_{\mathrm{s}} - \bar{s}_i^p) - c \leq 0 \tag{25}$$

$$\hat{s}_{\mathrm{s}} - \bar{s}_i^p \geq \frac{f'(\bar{s}_i^q) - c}{\beta} \tag{26}$$

□

In the case where local investment for an agent is equal in both layers, then any investing agent must invest equally in both layers.

Lemma 4. *Let s^* be an equilibrium in the network (N, g). For any agent i whose local investment is equal in both layers and less than $\hat{s}_{\mathrm{s}}$, that is $\bar{s}_i^1 = \bar{s}_i^2 < \hat{s}_{\mathrm{s}}$, i must be making investment $s_i = (\tilde{s}_i, \tilde{s}_i)$, where $f'(\tilde{s}_i + \bar{s}_i^1) - \beta\tilde{s}_i - c = 0$.*

Proof. Because $\bar{s}_i^1 = \bar{s}_i^2$, Equations (3) and (4) are equivalent, and a solution to one is a solution to both. Recall Equation (3):

$$f'(s_i^1 + \bar{s}_i^1) - \beta s_i^1 - c \leq 0 \tag{27}$$

Because $\bar{s}_i^1 < \hat{s}_\mathrm{s}$,

$$\begin{aligned} f'(0 + \bar{s}_i^1) - \beta 0 - c &= f'\left(\bar{s}_i^1 - c\right) \qquad (28)\\ &> f'(\hat{s}_\mathrm{s}) - c \qquad (29)\\ &> 0, \qquad (30) \end{aligned}$$

and because the left side of Equation (27) is continuous and decreasing in $\tilde{s}_{i1}$, there must be some $\tilde{s}_i > 0$ for which $f'(\tilde{s}_i + \bar{s}_i^1) - \beta \tilde{s}_i - c = 0$.

By the construction of $\tilde{S}$, any solution to Equation (27) may not exceed the upper boundary of $\tilde{S}$. Then $s_i = (\tilde{s}_i, \tilde{s}_i) \in \tilde{S}$ is a solution to Equations (3) and (4). Lemma 1 ensures that this is the unique solution to agent i's maximisation problem.

□

The following corollary is a direct result of Lemma 4.

Corollary 1. *Let $s^* \in S^n$ be an equilibrium of the network (N, g). If there exists an agent i who has no local investment in either layer, then i must be a dual-actor making investment $s_i^* = (\hat{s}_\mathrm{d}, \hat{s}_\mathrm{d})$, where $\hat{s}_\mathrm{d}$ is the solution to the equation $f'(\hat{s}_\mathrm{d}) - \beta \hat{s}_\mathrm{d} - c = 0$.*

Proof. From Lemma 4, we have that $\forall\, x \in [0, \hat{s}_\mathrm{s})$, Equations (3) and (4) have a unique symmetric solution, $s_i^* = (\tilde{s}_i, \tilde{s}_i)$, if $\bar{s}_i^1 = \bar{s}_i^2 = x$. Setting

$\bar{s}_i^1 = \bar{s}_i^2 = 0$, we find that $s_i^* = (\hat{s}_{\mathrm{d}}, \hat{s}_{\mathrm{d}})$, where $\hat{s}_{\mathrm{d}}$ is the solution to the equation $f'(\hat{s}_{\mathrm{d}}) - \beta \hat{s}_{\mathrm{d}} - c = 0$.

□

Lemmas 1 to 4 and Corollary 1 provide sufficient support for all of the claims in Proposition 1.

□

Proof of Proposition 2. The payoff of a dual-actor in a specialist equilibrium is $\Pi_d(s^* \mid g) = 2f(\hat{s}_{\mathrm{d}}) - 2c\hat{s}_{\mathrm{d}} - \beta \left(\hat{s}_{\mathrm{d}}\right)^2$. As well, since a free-rider receives payoff of at least $f(\hat{s}_{\mathrm{s}})$ from each layer, the payoff to a free-rider must satisfy $\Pi_\ell(s^* \mid g) \geq 2f(\hat{s}_{\mathrm{s}})$. Then, $\Pi_\ell(s^* \mid g) \geq 2f(\hat{s}_{\mathrm{s}}) > 2f(\hat{s}_{\mathrm{d}}) - 2c\hat{s}_{\mathrm{d}} - \beta \left(\hat{s}_{\mathrm{d}}\right)^2 = \Pi_d(s^* \mid g)$, and free-riders are always better off than dual-actors in specialised equilibria.

The payoff that a single-actor receives in the layer in which they are investing is $f(\hat{s}_{\mathrm{s}}) - c\hat{s}_{\mathrm{s}}$. In the other layer, a single-actor must be connected to at least one investing node, otherwise the single-actor would switch to dual-action. Thus, the payoff to a single-actor must satisfy

$$\begin{aligned} \Pi_i(s^* \mid g) &\geq \left[f(\hat{s}_{\mathrm{s}}) - c\hat{s}_{\mathrm{s}}\right] + f(\hat{s}_{\mathrm{d}}) && (31)\\ &> f(\hat{s}_{\mathrm{d}}) - c\hat{s}_{\mathrm{d}} + f(\hat{s}_{\mathrm{d}}) && (32)\\ &> 2f(\hat{s}_{\mathrm{d}}) - 2c\hat{s}_{\mathrm{d}} - \beta \left(\hat{s}_{\mathrm{d}}\right)^2 && (33)\\ &= \Pi_d(s^* \mid g) && (34) \end{aligned}$$

and the payoff of a single-actor must be greater than the payoff of a dual-actor. Step 32 is based on the fact that $\hat{s}_{\mathrm{s}}$ is the optimal investment amount for a single-actor, so $f(\hat{s}_{\mathrm{s}}) - c\hat{s}_{\mathrm{s}} > f(\hat{s}_{\mathrm{d}}) - c\hat{s}_{\mathrm{d}}$, with a strict inequality because f is strictly concave and thus $\hat{s}_{\mathrm{s}}$ is unique.

Thus, all dual-actors will receive payoff less than that of a single-actor or a free-rider.

$\square$

Proof of Theorem 2. First, we will prove that if N can be partitioned into the four disjoint sets, L, I^1, I^2, and D, as defined in Theorem 2, and that the agents take the action profile s such that: $s_\ell = (0,0)\;\forall\,\ell \in L$, $s_{i^1} = (\hat{s}_{\mathrm{s}}, 0)\;\forall\,i^1 \in I^1$, $s_{i^2} = (0, \hat{s}_{\mathrm{s}})\;\forall\,i^2 \in I^2$, and $s_d = (\hat{s}_{\mathrm{d}}, \hat{s}_{\mathrm{d}})\;\forall\,d \in D$, then this is a stable equilibrium.

Let k be the maximum degree of any agent in either g^1 or g^2.

Note that, because $f'(\hat{s}_{\mathrm{d}}) - \beta\hat{s}_{\mathrm{d}} - c = 0$, we know that $f'(\hat{s}_{\mathrm{d}}) - \beta\hat{s}_{\mathrm{s}} - c < 0$. As well, $f'(0) - \beta\hat{s}_{\mathrm{s}} - c > 0$, otherwise single-action could be a local maximum in autarky, which would violate Lemma 1. Because $f'(\hat{s}_{\mathrm{d}} - x) - \beta(\hat{s}_{\mathrm{s}} - x) - c$ is continuous and increasing in x, there must then be some $x \in (0, \hat{s}_{\mathrm{d}})$ such that $f'(\hat{s}_{\mathrm{d}} - x) - \beta(\hat{s}_{\mathrm{s}} - x) - c = 0$.

Following similar logic, because $f'(\hat{s}_{\mathrm{d}}) - \beta\hat{s}_{\mathrm{s}} - c < 0$ and $f'(0) - \beta\hat{s}_{\mathrm{s}} - c > 0$, there is some $y \in (0, \hat{s}_{\mathrm{d}})$ such that $f'(y) - \beta\hat{s}_{\mathrm{s}} - c = 0$. And finally, let $z = \hat{s}_{\mathrm{d}} - y$.

Let $\delta = \left\lfloor \frac{\hat{s}_{\mathrm{s}}}{\hat{s}_{\mathrm{d}}} + 1 \right\rfloor$, so δ is the minimum number of dual-actors that a free-rider who is not connected to any single-actor must be connected to, according to Item 2a. Assume that

$$\rho < \min\left\{\frac{\hat{s}_{\mathrm{d}}}{k}, \frac{\delta\hat{s}_{\mathrm{d}} - \hat{s}_{\mathrm{s}}}{\delta k}, \frac{x}{k}, \frac{y}{k}, \frac{z}{k}\right\} \tag{35}$$

. Let $\boldsymbol{\epsilon}$ be any $n \times 2$ vector such that $|\epsilon_i^p| < \rho$ and $s_i^p + \epsilon_i^p \geq 0$, $\forall\, i \in N$, $p \in \{1, 2\}$.

Given a sequence of best responses to starting vector s, let $r_i^{p,t}(s \mid g)$ denote i's investment in layer p in the t^{th} element of the sequence.

Step 1 : Consider $r^1(s+\boldsymbol{\epsilon} \mid g)$.

In layer g^p, for any $\ell \in L$, if condition 2a holds, then ℓ is connected to at least δ agents in D. After a permutation of s by $\boldsymbol{\epsilon}$, each agent's investment may be reduced by at most ρ. Thus, after such a permutation,

$$\begin{aligned}\bar{s}^p_\ell &\geq \delta \hat{s}_{\mathrm{d}} - k\rho && (36)\\ &> \delta \hat{s}_{\mathrm{d}} - k\frac{\delta \hat{s}_{\mathrm{d}} - \hat{s}_{\mathrm{s}}}{\delta k} && (37)\\ &> \delta \hat{s}_{\mathrm{d}} - k\frac{\delta \hat{s}_{\mathrm{d}} - \hat{s}_{\mathrm{s}}}{k} && (38)\\ &= \hat{s}_{\mathrm{s}}. && (39)\end{aligned}$$

Agent ℓ will therefore make no investment in the first step after a permutation by $\boldsymbol{\epsilon}$.

Alternatively, suppose that Item 2b holds, and ℓ is connected to at least one agent in I^p and more than one agent in $D \cup I^p$. Then after a permutation of s by $\boldsymbol{\epsilon}$, ℓ's local investment will be such that

$$\begin{aligned}\bar{s}^p_\ell &\geq \hat{s}_{\mathrm{s}} + \hat{s}_{\mathrm{d}} - k\rho && (40)\\ &> \hat{s}_{\mathrm{s}} + \hat{s}_{\mathrm{d}} - k\frac{\hat{s}_{\mathrm{d}}}{k} && (41)\\ &= \hat{s}_{\mathrm{s}}. && (42)\end{aligned}$$

Again, agent ℓ will make no investment in the first step after a permutation by $\boldsymbol{\epsilon}$. We may thus conclude that $r^1_\ell(s+\boldsymbol{\epsilon} \mid g) = (0,0)$.

In layer g^1, any $i^2 \in I^2$ is connected to at least one agent in $D \cup I^1$. After a permutation of s by $\boldsymbol{\epsilon}$, it is then the case that $\bar{s}^1_{i^2} \geq \hat{s}_\text{d} - k\rho$. In layer g^2, i^2 will have previously had no investment of effort by neighbours, and thus permutation by $\boldsymbol{\epsilon}$ results in $\bar{s}^2_{i^2} \leq k\rho$. This, in turn, implies that i^2's optimal single-action investment in g^2 is weakly greater than $\hat{s}_\text{s} - k\rho$. Then, i^2's marginal payoff from investment in g^1 would be

$$\frac{\partial \Pi_{i^2}(s + \boldsymbol{\epsilon} \mid g)}{\partial s^1_{i^2}} \leq f'(\hat{s}_\text{d} - k\rho) - \beta(\hat{s}_\text{s} - k\rho) - c \tag{43}$$

$$< f'(\hat{s}_\text{d} - k\frac{x}{k}) - \beta(\hat{s}_\text{s} - k\frac{x}{k}) - c \tag{44}$$

$$= f'(\hat{s}_\text{d} - x) - \beta(\hat{s}_\text{s} - x) - c \tag{45}$$

$$= 0 \tag{46}$$

By Lemma 3, we may then conclude that in the first step after permutation by $\boldsymbol{\epsilon}$, any agent i^2 will be a single-actor investing in layer g^2, and that i^2's investment in g^2 will be at least $\hat{s}_\text{s} - k\rho$. That is, $r^1_{i^2}(s + \boldsymbol{\epsilon} \mid g) = (0, \tilde{s}_{i^2})$, where $\tilde{s}_{i^2} \in [\hat{s}_\text{s} - k, \hat{s}]$ may vary for each member of I^2.

By symmetry, we may conclude as well that $r^1_{i^1}(s + \boldsymbol{\epsilon} \mid g) = (\tilde{s}_{i^1}, 0)$, where $\tilde{s}_{i^1} \in [\hat{s}_\text{s} - k, \hat{s}]$ may vary for each member of I^1.

In layer g^1, any $d \in D$ has no neighbours in $D \cup I^1$. After a permutation of s by $\boldsymbol{\epsilon}$, we know that $\bar{s}^p_d \in [0, k\rho]$. We wish to consider whether it is possible for d to switch to single action. Suppose that d were to become a single-actor in layer g^p. Then,

$$\frac{\partial \Pi_d(s+\boldsymbol{\epsilon} \mid g)}{\partial s_d^1} \geq f'(k\rho) - \beta \hat{s}_{\mathrm{s}} - c \tag{47}$$

$$> f'(k\frac{y}{k}) - \beta \hat{s}_{\mathrm{s}} - c \tag{48}$$

$$= f'(y) - \beta \hat{s}_{\mathrm{s}} - c \tag{49}$$

$$= 0. \tag{50}$$

By Lemma 3, d cannot be a single-actor, and must invest in both layers after a permutation of s by $\boldsymbol{\epsilon}$. If d, reduces investment in layer g^p, then marginal payoff increases in layer g^q, which would incentivise d to increase investment in layer g^q. Thus, $\hat{s}_{\mathrm{d}} - k\rho$ represents the minimal amount that d could invest in either layer, and we may write that $r_d^1(s+\boldsymbol{\epsilon} \mid g) = (\tilde{s}_d^1, \tilde{s}_d^2)$, where $\tilde{s}_d^1, \tilde{s}_d^2 \in [\hat{s}_{\mathrm{d}} - k\rho, \hat{s}_{\mathrm{d}}]$ and both values may vary for each member of D. We will next consider what investment any agent will make after two steps of myopic best responses to the permutation $\boldsymbol{\epsilon}$.

Step 2 : Consider $r^2(\boldsymbol{s}+\boldsymbol{\epsilon}|g)$.

In Step 1, we showed that all free-riders, any agent $\ell \in L$, will make no investment, and single-actors will not invest in the layer in which they were not investing initially. Since dual-actors may only have connections to these two types of agents, we can conclude that after the first step, $\bar{s}_d^p = 0 \, \forall \, p \in \{1, 2\}$. Thus, we conclude that dual-agents will return to their original investments in the second step of myopic best responses, that is $r_d^2(s+\boldsymbol{\epsilon} \mid g) = (\hat{s}_{\mathrm{d}}, \hat{s}_{\mathrm{d}})$. Consider any $i^1 \in I^1$. In g^1, i^1 may only be connected to agents in L and I^2.

Since none of these agents will invest in layer g^1 in step 1, in step 2 $\bar{s}_{i^1 1} = 0$, and i^1's optimal single-action investment in g^1 is $\hat{s}_{\mathrm{s}}$. Now, suppose that i^1 does make this investment. i^1 is connected to at least one member of $D \cup I^2$ in g^2, and therefore $\bar{s}^2_{i^1} \leq \hat{s}_{\mathrm{d}} - k\rho$. i^1's marginal payoff in g^2 is

$$\frac{\partial \Pi_{i^1}(s_{Step1} \mid g)}{\partial s^2_{i^1}} \leq f'(\hat{s}_{\mathrm{d}} - k\rho) - \beta \hat{s}_{\mathrm{s}} - c \tag{51}$$

$$< f'(\hat{s}_{\mathrm{d}} - k\frac{z}{k}) - \beta \hat{s}_{\mathrm{s}} - c \tag{52}$$

$$= f'(y) - \beta \hat{s}_{\mathrm{s}} - c \tag{53}$$

$$= 0. \tag{54}$$

Thus, $r^2_{i^1}(s + \boldsymbol{\epsilon} \mid g) = (\hat{s}_{\mathrm{s}}, 0)$ satisfies agent i's first order conditions, and by Lemma 1, we may conclude that any agent $i^1 \in I^1$ will make this investment.

By symmetry, we conclude as well that $r^2_{i^2}(s + \boldsymbol{\epsilon} \mid g) = (0, \hat{s}_{\mathrm{s}})$.

Finally, consider any agent $\ell \in L$. If Item 2b holds, then ℓ will be connected to at least one investing single-actor and at least one dual-actor in either layer g^p. Then

$$\bar{s}^p_\ell \geq (\hat{s}_{\mathrm{s}} - k\rho) + (\hat{s}_{\mathrm{d}} - k\rho) \tag{55}$$

$$> \hat{s}_{\mathrm{s}} - \hat{s}_{\mathrm{d}} - 2k\frac{\hat{s}_{\mathrm{d}}}{k} \tag{56}$$

$$= \hat{s}_{\mathrm{s}} + \hat{s}_{\mathrm{d}} \tag{57}$$

$$> \hat{s}_{\mathrm{s}}. \tag{58}$$

So ℓ will not invest in either layer.

Now, consider the case when Item 2a holds, then ℓ is connected to at least δ

dual-actors in either layer g^p. In this case,

$$\bar{s}^p_\ell \geq \delta(\hat{s}_\mathrm{d} - k\rho) \tag{59}$$

$$> \delta\hat{s}_\mathrm{d} - \delta k \frac{\delta\hat{s}_\mathrm{d} - \hat{s}_\mathrm{s}}{\delta k} \tag{60}$$

$$= \hat{s}_\mathrm{s}. \tag{61}$$

Thus, ℓ will not invest in either layer. We may then conclude that $r^2_\ell(s + \boldsymbol{\epsilon} \mid g) = (0, 0)$.

Thus, we have shown that all agents will return to their original investments after a permutation of s by $\boldsymbol{\epsilon}$ and two steps of myopic best responses.

Now, we will prove the opposite direction, that an equilibrium that is stable must be characterised as set out in Theorem 2. This requires the following lemma.

Lemma 5. *Let $s, s' \in S$ be two distinct action profiles such that $\forall\, i \in N$ $s^1_i \geq s^{1\prime}_i$, and $s^2_i \leq s^{2\prime}_i$. Then, $\forall\, i \in N$, $r^{1,2}_i(s \mid g) \geq r^{1,2}_i(s' \mid g)$ and $r^{2,2}_i(s \mid g) \leq r^{2,2}_i(s' \mid g)$.*

As this lemma is a notationally dense, we will state it in words. Given the two action profiles $s, s' \in S$, if the actions of all agents in layer g^1 are weakly greater in s than in s', and the actions of all agents in layer g^2 are weakly greater in s' than in s, then after two steps of simultaneous myopic best responses by all agents the same relationships will hold.

Proof of Lemma 5. Let $s, s' \in S$ be two separate action profiles such that $\forall\, i \in N$, $s^1_i \geq s^{1\prime}_i$ and $s^2_i \leq s^{2\prime}_i$.

Pick any arbitrary agent $i \in N$, and suppose that $r^1_i(s) = (\hat{s}_\mathrm{s} - \bar{s}^1_i, 0)$. Then we know that $f'(\bar{s}_i^2) - \beta(\hat{s}_\mathrm{s} - \bar{s}_i^1) - c \leq 0$ by Lemma 3. But then

$$f'\left(\bar{s}_i^{2\prime}\right) - \beta\left(\hat{s}_\mathrm{s}' - \bar{s}_i^{1\prime}\right) - c \leq f'\left(\bar{s}_i^2\right) - \beta\left(\hat{s}_\mathrm{s} - \bar{s}_i^1\right) - c \tag{62}$$

$$\leq 0, \tag{63}$$

so $r_i^1(s') = \left(\hat{s}_\mathrm{s}' - \bar{s}_i^{1\prime}, 0\right)$, and because $\bar{s}_i^{1\prime} \leq \bar{s}_i^{1\prime}$, i's investment in g^1 in $r_i^1(s')$ must be weakly larger than i's investment in $r_i^1(s)$, and thus $r_i^{1,1}(s) \leq r_i^{1,1}(s')$ and $r_i^{2,1}(s) \geq r_i^{2,1}(s')$.

By a similar argument, it can be shown that if $r_i^1(s') = \left(0, \hat{s}_\mathrm{s} - \bar{s}_i^{2\prime}\right)$, then it must be the case that $r_i^1(s) = (0, \hat{s}_\mathrm{s} - \bar{s}_i^2)$, and so $r_i^{1,1}(s) \leq r_i^{1,1}(s')$ and $r_i^{2,1}(s) \geq r_i^{2,1}(s')$.

Because a single-actor will always have a greater marginal payoff function in the layer of investment than a dual-actor, if $r_i^1(s') = \left(\hat{s}_\mathrm{s}' - \bar{s}_i^{1\prime}, 0\right)$ and $r_i^1(s)$ involves dual-investment, or if $r_i^1(s) = (0, \hat{s}_\mathrm{s} - \bar{s}_i^2)$ and $r_i^1(s')$ involves dual-investment, then the relationships $r_i^{1,1}(s) \leq r_i^{1,1}(s')$ and $r_i^{2,1}(s) \geq r_i^{2,1}(s')$ must hold.

Suppose that $r_i(s) = (0, \hat{s}_\mathrm{s} - \bar{s}_i^2)$ and $r_i(s') = (\hat{s}_\mathrm{s} - \bar{s}_i^{1\prime}, 0)$. Then clearly $r_i^{1,1}(s) \leq r_i^{1,1}(s')$ and $r_i^{2,1}(s) \geq r_i^{2,1}(s')$.

We have now shown for all boundary solutions that $r_i^{1,1}(s) \leq r_i^{1,1}(s')$ and $r_i^{2,1}(s) \geq r_i^{2,1}(s')$, which leaves only the case when both $r_i^1(s)$ and $r_i^1(s')$ involve dual-investment.

We know that the system

$$f'(\bar{x} + x) - \beta y - c = 0 \text{ and} \tag{64}$$

$$f'(\bar{y} + y) - \beta x - c = 0 \tag{65}$$

is unique, due to Lemma 1. Now we consider what happens to the solution when $\bar{x}$ changes and $\bar{y}$ remains constant. From Equation (65),

$$f''(\bar{y}+y)\,\mathrm{d}y - \beta\,\mathrm{d}x = 0 \tag{66}$$

$$\mathrm{d}y = \frac{\beta}{f''(\bar{y}+y)}\,\mathrm{d}x. \tag{67}$$

Then plug this into the total derivative of Equation (64)

$$f''(\bar{x}+x)(\mathrm{d}\bar{x}+\mathrm{d}x) - \beta\frac{\beta}{f''(\bar{y}+y)}\,\mathrm{d}x = 0 \tag{68}$$

$$f''(\bar{x}+x)\left(1+\frac{\mathrm{d}x}{\mathrm{d}\bar{x}}\right) - \frac{\beta^2}{f''(\bar{y}+y)}\frac{\mathrm{d}x}{\mathrm{d}\bar{x}} = 0 \tag{69}$$

$$\frac{\mathrm{d}x}{\mathrm{d}\bar{x}} = \frac{-f''(\bar{x}+x)f''(\bar{y}+y)}{f''(\bar{x}+x)f''(\bar{y}+y)-\beta^2}. \tag{70}$$

Because of Assumption 1, we know that the denominator of Equation (70) is positive, while the numerator is negative. Thus, we may conclude that, at the unique solution, $\frac{\mathrm{d}x}{\mathrm{d}\bar{x}} < 0$, which in turn implies that $\frac{\mathrm{d}y}{\mathrm{d}\bar{x}} > 0$, from Equation (67). If $\bar{x}$ increases and $\bar{y}$ decreases, sequential application of this conclusion ensures that the optimal value of x will decrease and the optimal value of y will increase.

We know that $\bar{s}_i^1 \geq \bar{s}_i^{1\prime}$ and $\bar{s}_i^2 \leq \bar{s}_i^{2\prime}\;\forall\, i \in N$. If both $r_i^1(s)$ and $r_i^1(s')$ involve dual-investment, then from Equations (67) and (70) we may conclude that $r_i^{1,1}(s) \leq r_i^{1,1}(s')$ and $r_i^{2,1}(s) \geq r_i^{2,1}(s')$, and we can therefore conclude that these two relationships always hold.

But then, by applying this fact twice, it must be true that a second step of best-response function r yields $r_i^{1,2}(s) \geq r_i^{1,2}(s')$ and $r_i^{2,2}(s) \leq r_i^{2,2}(s')\;\forall\, i \in N$, which proves this lemma.

□

Now, suppose that $s^* \in S$ is a stable equilibrium, and suppose there exists an investor i such that $\bar{s}_i^1 \leq \bar{s}_i^2 < \hat{s}_{\mathrm{s}}$, and $\hat{s}_{\mathrm{s}} - \bar{s}_i^1 < \frac{f'(\bar{s}_i^2)-c}{\beta}$. Proposition 1 tells us that i will be making an interior investment in both layers. Let i's investment be $s_i = (s_i^1, s_i^2)$. Because i's investment is interior, there must be some $\delta > 0$ such that $\tilde{s}_i = (s_i^1 + \delta, s_i^2) \in \tilde{S}$. Now, suppose s^* is permuted such that i now invests $\tilde{s}_i$ and no other agent changes their investment. By Lemma 5, in every second iteration of the best response function, i's investment in g^1 is greater than $s_i^1 + \delta$. Therefore, a sequence of best responses to this permutation will not converge to s^*. Thus, there can be no agent i making an interior investment in both layers.

Now, suppose there exists an investor i such that $\bar{s}_i^1 \leq \bar{s}_i^2 < \hat{s}_{\mathrm{s}}$, and $\hat{s}_{\mathrm{s}} - \bar{s}_i^1 \geq \frac{f'(\bar{s}_i^2)-c}{\beta}$. Proposition 1 tells us that i will be making an investment only in layer g^1. Let i's investment be $s_i = (s_i^1, 0)$, with $s_i^1 < \hat{s}_{\mathrm{s}}$. Then, there must be some $\delta > 0$ such that $s_i^1 + \delta < \hat{s}_{\mathrm{s}}$. Now, suppose s^* is permuted such that i now invests makes investment $(s_i^1 + \delta, 0)$ and no other agent changes their investment. By Lemma 5, in every second iteration of the best response function, i's investment in g^1 is greater than $s_i^1 + \delta$. Therefore, a sequence of best responses to this permutation will not converge to s^*. Thus, there can be no agent i making investment s_i.

We have shown that there may be no interior investors in a stable equilibrium, leaving only specialists investing $(0,0)$, $(\hat{s}_{\mathrm{s}}, 0)$, $(0, \hat{s}_{\mathrm{s}})$, or $(\hat{s}_{\mathrm{d}}, \hat{s}_{\mathrm{d}})$. According to the criteria established in Proposition 1, such investments will only be best responses with the agents may be partitioned into the sets L, I^1, I^2, and D.

□

Proof of Proposition 3. I have claimed that the function

$$\phi(s) = \begin{pmatrix} \alpha^1 \mathbf{1} \\ \alpha^2 \mathbf{1} \end{pmatrix}^{\mathsf{T}} s - \frac{1}{2} s^{\mathsf{T}} \left(\boldsymbol{I} - \begin{bmatrix} \delta^1 \boldsymbol{G}^1 & \beta \boldsymbol{I} \\ \beta \boldsymbol{I} & \delta^2 \boldsymbol{G}^2 \end{bmatrix} \right) s \tag{71}$$

satisfies the properties of a potential function. That is, $\forall\, s_i, s_i'$ and $\forall\, i \in N$,

$$\phi(s_i, s_{-i}) - \phi\left(s_i', s_{-i}\right) = \Pi_i(s_i, s_{-i} \mid g) - \Pi_i\left(s_i', s_{-i} \mid g\right). \tag{72}$$

Then, $\forall\, s, s' \in S$ such that for any $i \in N$, $s_i \neq s_i'$ implies $s_j = s_j' \,\forall\, j \neq i$,

$$
\begin{aligned}
&\phi(s_i, s_{-i}) - \phi\left(s_i', s_{-i}\right) && (73)\\
=&\phi(s) - \phi\left(s'\right) && (74)\\
=&\begin{pmatrix}\alpha^1\mathbf{1}\\ \alpha^2\mathbf{1}\end{pmatrix}^{\mathsf{T}} s - \frac{1}{2}s^{\mathsf{T}}\left(\boldsymbol{I} - \begin{bmatrix}\delta^1\boldsymbol{G}^1 & \beta\boldsymbol{I}\\ \beta\boldsymbol{I} & \delta^2\boldsymbol{G}^2\end{bmatrix}\right)s\\
&\quad - \begin{pmatrix}\alpha^1\mathbf{1}\\ \alpha^2\mathbf{1}\end{pmatrix}^{\mathsf{T}} s' - \frac{1}{2}s'^{\mathsf{T}}\left(\boldsymbol{I} - \begin{bmatrix}\delta^1\boldsymbol{G}^1 & \beta\boldsymbol{I}\\ \beta\boldsymbol{I} & \delta^2\boldsymbol{G}^2\end{bmatrix}\right)s' && (75)\\
=&\sum_{j\in N}\left[\alpha^1 s_j^1 + \alpha^2 s_j^2 - \frac{1}{2}\left(\left(s_j^1\right)^2 + \left(s_j^2\right)^2\right) + \beta s_j^1 s_j^2\right]\\
&\quad + \frac{1}{2}\left[\sum_{j\in N}\sum_{k\in N}(\delta^1 g_{jk}^1 s_j^1 s_k^1 + \delta^2 g_{jk}^2 s_j^2 s_k^2)\right]\\
&\quad - \left\{\sum_{j\in N}\left[\alpha^1 s_j^{1\prime} + \alpha^2 s_j^{2\prime} - \frac{1}{2}\left(\left(s_j^{1\prime}\right)^2 + \left(s_j^{2\prime}\right)^2\right) + \beta s_j^{1\prime} s_j^{2\prime}\right]\right.\\
&\quad \left. + \frac{1}{2}\left[\sum_{j\in N}\sum_{k\in N}(\delta^1 g_{jk}^1 s_j^{1\prime} s_k^{1\prime} + \delta^2 g_{jk}^2 s_j^{2\prime} s_k^{2\prime})\right]\right\} && (76)\\
=&\,\alpha^1 s_i^1 + \alpha^2 s_i^2 - \frac{1}{2}\left(\left(s_i^1\right)^2 + \left(s_i^2\right)^2\right) + \beta s_i^1 s_i^2 + \delta^1 s_i^1 \sum_{j\in N} g_{ij}^1 s_j^1 + \delta^2 s_i^2 \sum_{j\in N} g_{ij}^2 s_j^2\\
&\quad - \left[\alpha^1 s_i^{1\prime} + \alpha^2 s_i^{2\prime} - \frac{1}{2}\left(\left(s_i^{1\prime}\right)^2 + \left(s_i^{2\prime}\right)^2\right) + \beta s_i^{1\prime} s_i^{2\prime}\right.\\
&\quad \left. + \delta^1 s_i^{1\prime}\sum_{j\in N} g_{ij}^1 s_j^{1\prime} + \delta^2 s_i^{2\prime}\sum_{j\in N} g_{ij}^2 s_j^{2\prime}\right] && (77)\\
=&\Pi_i(s_i, s_{-i} \mid g) - \Pi_i\left(s_i', s_{-i} \mid g\right) && (78)
\end{aligned}
$$

Thus, ϕ satisfies the properties of a potential function.

$\square$

Proof of Theorem 3. Let

$$\upsilon^p = \begin{cases} \delta^p \lambda_{\max}(\boldsymbol{G}^p) & \text{if } \delta^p > 0 \\ \delta^p \lambda_{\min}(\boldsymbol{G}^p) & \text{if } \delta^p < 0. \end{cases} \tag{79}$$

If

$$\beta^2 < (1-\upsilon^1)(1-\upsilon^2), \tag{80}$$

then the game $\Gamma(N, S, g)$ has a unique equilibrium on the action space S.

From Proposition 3, we know that the potential function for the game $\Gamma(N, S, g)$ is

$$\phi(s) = \begin{pmatrix} \alpha^1 \mathbf{1} \\ \alpha^2 \mathbf{1} \end{pmatrix}^{\mathsf{T}} s - \frac{1}{2} s^{\mathsf{T}} \left(\boldsymbol{I} - \begin{bmatrix} \delta^1 \boldsymbol{G}^1 & \beta \boldsymbol{I} \\ \beta \boldsymbol{I} & \delta^2 \boldsymbol{G}^2 \end{bmatrix} \right) s. \tag{81}$$

Then,

$$\frac{\partial \phi(s)}{\partial s} = \begin{pmatrix} \alpha^1 \mathbf{1} \\ \alpha^2 \mathbf{1} \end{pmatrix} - \left(\boldsymbol{I} - \begin{bmatrix} \delta^1 \boldsymbol{G}^1 & \beta \boldsymbol{I} \\ \beta \boldsymbol{I} & \delta^2 \boldsymbol{G}^2 \end{bmatrix} \right) s \tag{82}$$

and

$$\nabla \phi(s) = - \left(\boldsymbol{I} - \begin{bmatrix} \delta^1 \boldsymbol{G}^1 & \beta \boldsymbol{I} \\ \beta \boldsymbol{I} & \delta^2 \boldsymbol{G}^2 \end{bmatrix} \right). \tag{83}$$

To prove that $\phi(s)$ has a unique global maximum on S, it is sufficient to show that $-\nabla \phi(s)$ is positive definite on S. First, let

$$\boldsymbol{M} = -\nabla \phi(s) = \begin{bmatrix} \boldsymbol{I} - \delta^1 \boldsymbol{G}^1 & -\beta \boldsymbol{I} \\ -\beta \boldsymbol{I} & \boldsymbol{I} - \delta^2 \boldsymbol{G}^2 \end{bmatrix}. \tag{84}$$

$\boldsymbol{M}$ is positive definite if and only if its upper left block and the Schur complement if its upper left block are positive definite. These are $(\boldsymbol{I} - \delta^1 \boldsymbol{G}^1)$ and $(\boldsymbol{I} - \delta^2 \boldsymbol{G}^2) - \beta^2 (\boldsymbol{I} - \delta^1 \boldsymbol{G}^1)^{-1}$ respectively.

Recall that

$$\upsilon^p = \begin{cases} \delta^p \lambda_{\max}(\boldsymbol{G}^p) & \text{if } \delta^p > 0 \\ \delta^p \lambda_{\min}(\boldsymbol{G}^p) & \text{if } \delta^p < 0. \end{cases} . \tag{85}$$

Thus, $-\upsilon^1$ is the minimum eigenvalue of $-\delta^1 \boldsymbol{G}^1$, which requires the fact that the minimum eigenvalue of an adjacency matrix is less than or equal to 0. It follows, then, that $1 - \upsilon^1$ is the minimum eigenvalue of $(\boldsymbol{I} - \delta^1 \boldsymbol{G}^1)$. We may thus conclude that $(\boldsymbol{I} - \delta^1 \boldsymbol{G}^1)$ is positive definite if and only if $1 - \upsilon^1 > 0$.

Consider now the Schur complement, $(\boldsymbol{I} - \delta^2 \boldsymbol{G}^2) - \beta^2 (\boldsymbol{I} - \delta^1 \boldsymbol{G}^1)^{-1}$. By Weyl's inequality

$$\lambda_{\min}\left[(\boldsymbol{I} - \delta^2 \boldsymbol{G}^2) - \beta^2 (\boldsymbol{I} - \delta^1 \boldsymbol{G}^1)^{-1}\right] \tag{86}$$

$$\geq \lambda_{\min}(\boldsymbol{I} - \delta^2 \boldsymbol{G}^2) + \lambda_{\min}(-\beta^2 (\boldsymbol{I} - \delta^1 \boldsymbol{G}^1)^{-1}) \tag{87}$$

$$= (1 - \upsilon^2) - \frac{\beta^2}{1 - \upsilon^1}. \tag{88}$$

Now, we have assumed

$$\beta^2 < (1 - \upsilon^1)(1 - \upsilon^2) \tag{89}$$

$$\iff (1 - \upsilon^2) - \frac{\beta^2}{1 - \upsilon^1} > 0, \tag{90}$$

which is sufficient to show that the minimum eigenvalue of the Schur complement is greater than zero, and thus the Schur complement is positive definite.

Thus, the assumptions that $1 - \upsilon^1 > 0$ and $\beta^2 < (1 - \upsilon^1)(1 - \upsilon^2)$ are sufficient to ensure that there is a unique equilibrium.

$\square$