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Guilherme Carmona
University of Surrey

Hamid Sabourian
University of Cambridge

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Guilherme Carmona
University of Surrey

Hamid Sabourian
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Abstract

We establish a version of Blackwell's (1956) approachability result with discounting. Our main result shows that, for convex sets, our notion of approachability with discounting is equivalent to Blackwell's (1956) approachability. Our proofs are based on a concentration result for probability measures and on the minmax theorem for two-person, zero-sum games.

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1 Introduction

Blackwell's (1956) approachability result gives, in repeated two-player games with vector payoffs, a necessary and sufficient condition for each player to be able to guarantee that the average payoff is, with high probability and after sufficiently periods have passed, close to a given convex set, independently of the strategy of the other player and of the time period. Several extensions and variations of this result have been given,[1] but always for the case here payoffs are not discounted, i.e. the average payoff is the arithmetic average of the payoffs received in the first n periods.[2]

The importance of no-discounting for Blackwell's (1956) approachability result is that this feature is critical for the use of the strong law of large numbers for martingales, i.e. the applicability of this result does not extend to the case of discounting (where the average payoff is the discounted average of the payoffs received in the first n periods). Extending Blackwell's (1956) approachability result to the discounted case thus requires a different approach.

In this paper, we provide a version of Blackwell's (1956) approachability result for the case of discounting. We show that the necessary and sufficient condition for a convex set to be approachable is also necessary and sufficient for a convex set to be approachable with discounting.

Our proof is based on the following elementary ideas. First, McDiarmid's (1998) concentration result allows us to reduce the problem to showing that the expected discounted average of payoffs (as opposed to the discounted average of realized payoffs) is close to a given set. Second, player 1 (say) can always make sure that the expected discounted average of payoffs belongs to a given convex set. This would give our result except for the fact that, in general, player 1's strategy depends on player 2's

[1] See, among others, Hou (1969), Hou (1971), Vieille (1992), Spinat (2002), Lehrer and Solan (2009), Shani and Solan (2014), Bauso, Lehrer, Solan, and Venel (2015), Lagziel and Lehrer (2015), Perchet and Quincampoix (2015) and Fournier, Kuperwasser, Munk, Solan, and Weinbaum (2020).

[2] An exception is provided by Vieille (1992) (see its concluding remarks), who has shown that every set is either weakly approachable or weakly excludable, both under discounting and no-discounting. See also Flesch, Laraki, and Perchet (2018), who consider weak approachability in quitting games.

strategy. However, the independence of player 1's strategy from that of player 2 can be achieved by applying the minmax theorem to the two-person, zero-sum game where player 1 seeks to minimize the expected distance of the discounted payoff to the given convex set.

Blackwell's (1956) approachability has found several applications in game theory, namely on repeated two-person, zero-sum games (see, e.g., Zamir (1992)) and on finitely repeated games with no discounting (see Gossner (1995)). In Barlo, Carmona, and Sabourian (2016), we have established a perfect monitoring Folk Theorem with bounded memory strategies in infinitely repeated games with discounting using a particular case of the main result established in the current paper. The approachability results of this paper can also be used to (i) obtain a perfect monitoring Folk Theorem with mixed strategies and finite automata in infinitely repeated games with discounting and 2-players and (ii) obtain the punishment strategies in Hörner and Lovo (2009).[3]

2 Approachability with discounting

We consider a setting similar to that of Blackwell (1956). There are two players, 1 and 2, who interact in every period $t \in \mathbb{N} = \{1, 2, \ldots\}$. In every such period, player 1 chooses an action from a finite set $A_1 = \{1, \ldots, r\}$, with $r \in \mathbb{N}$, and player 2 from $A_2 = \{1, \ldots, s\}$ with $s \in \mathbb{N}$. Players are allowed to randomize, i.e. choose elements of $\Delta(A_1)$ and $\Delta(A_2)$ respectively.[4] As in Blackwell (1956), let $P = \Delta(A_1)$ and $Q = \Delta(A_2)$.

Each player observes neither the mixed choice made by the other player nor (necessarily) the realization. Instead, both players observe a public signal from a finite

[3]The proofs of these claims can be found in Sections B and C respectively. For the case of more than two players Claim (i) regarding the Folk Theorem with mixed strategies also follows from Barlo, Carmona, and Sabourian's (2016) result with bounded recall; however, not only appealing to our results in this paper makes the proof more direct, it also works for the case with two players.

[4]Throughout this paper $\Delta(Y)$ will stand for the set of probability distributions over Y, when Y is a finite set.

subset X of $\mathbb{R}^N$, with $N \in \mathbb{N}$.[5] For each $i \in A_1$ and $j \in A_2$, let $m(i,j) \in \Delta(X)$ be the probability distribution on X when player 1 chooses i and player 2 chooses j. We let $m_x(i,j)$ denote the probability that the signal x is observed when players choose (i,j), for each $x \in X$. As in Blackwell (1956), M denotes the $r \times s$ matrix with generic element $m(i,j)$, with $1 \leq i \leq r$ and $1 \leq j \leq s$.

For any $t \geq 1$, a t-stage public history is a sequence $h = (x_1, \ldots, x_t) \in X^t$ (the t-fold Cartesian product of X). The set of all t-stage public histories is denoted by $H_t = X^t$. We represent the initial (empty) public history by $\emptyset$ and let $H_0 = \{\emptyset\}$. The set of all public histories is defined by $H = \bigcup_{t \in \mathbb{N}_0} H_t$.

A (behavior, public) *strategy* for player $i \in \{1, 2\}$ is a function $f_i : H \to \Delta(A_i)$ mapping public histories into mixed actions. The set of player i's strategies is denoted by F_i, and $F = F_1 \times F_2$. We let f denote a generic element of F_1 and g a generic element of F_2. Given $(f,g) \in F$, $h \in H$, $i \in A_1$ and $j \in A_2$, $f_i(h)$ denotes the probability that action i is played by player 1 and $g_j(h)$ denotes the probability that action j is played by player 2.

Given a strategy $(f,g) \in F$, for each $t \in \mathbb{N}$, $x \in X$ and $(x_1, \ldots, x_t) \in X^t$, let

$$\beta_x(x_1, \ldots, x_t; f, g) = \sum_{i=1}^{r} \sum_{j=1}^{s} f_i(x_1, \ldots, x_t) g_j(x_1, \ldots, x_t) m_x(i,j)$$

be the probability of x after public history $(x_1, \ldots, x_t)$ has occurred. Furthermore, let $\beta_x(\emptyset; f, g) = \sum_{i=1}^{r} \sum_{j=1}^{s} f_i(\emptyset) g_j(\emptyset) m_x(i,j)$. When it is clear from the context what the strategy (f,g) is, we simplify the notation and write $\beta_x(x_1, \ldots, x_t)$ instead of $\beta_x(x_1, \ldots, x_t; f, g)$.

A strategy $(f,g) \in F$ induces, for each $t \in \mathbb{N}$, a probability measure $P_{(f,g),t}$ on X^t and a probability measure $\pi_{(f,g),t}$ on X as follows. For each $x \in X = X^1$, $P_{(f,g),1}(x) = \pi_{(f,g),1}(x) = \beta_x(\emptyset)$. Assuming that $P_{(f,g),1}, \pi_{(f,g),1}, \ldots, P_{(f,g),t-1}, \pi_{(f,g),t-1}$ have been defined, then

$$P_{(f,g),t}(x_1, \ldots, x_t) = P_{(f,g),t-1}(x_1, \ldots, x_{t-1}) \beta_{x_t}(x_1, \ldots, x_{t-1})$$

[5]We depart from Blackwell (1956) in the assumption that X is finite since, in that paper, X is assumed to be a compact convex subset of $\mathbb{R}^N$.

for all $(x_1, \ldots, x_t) \in X^t$ and

$$\pi_{(f,g),t}(x) = \sum_{(x_1,\ldots,x_{t-1}) \in X^{t-1}} P_{(f,g),t}(x_1, \ldots, x_{t-1}, x)$$

for each $x \in X$. Moreover, $(f, g) \in F$ induces a probability measure $P_{(f,g),\infty}$ on X^∞ in the standard way, namely, such that

$$P_{(f,g),\infty}(E \times X \times X \times \cdots) = P_{(f,g),t}(E) \tag{1}$$

for each $E \subseteq X^t$.

Let $||\cdot||$ denote a norm on $\mathbb{R}^N$ and, for each $x \in \mathbb{R}^N$ and $S \subseteq \mathbb{R}^N$, let $d(x, S) = \inf_{y \in S} ||x - y||$. We now recall Blackwell's (1956) notion of approachability.

Definition 1 (Approachability) *A subset S of $\mathbb{R}^N$ is* approachable *if there exists $f \in F_1$ such that, for every $\varepsilon > 0$, there exists $T \in \mathbb{N}$ such that, for every $g \in F_2$,*

$$P_{(f,g),\infty}\left(\left\{(x_1, x_2, \ldots) \in X^\infty : d\left(\frac{1}{t}\sum_{k=1}^{t} x_k, S\right) \geq \varepsilon \text{ for some } t \geq T\right\}\right) < \varepsilon.$$

Our notion of approachability with discounting is as follows.

Definition 2 (Approachability with discounting) *A subset S of $\mathbb{R}^N$ is* approachable with discounting *if, for each $\delta \in (0, 1)$, there exists $f_\delta \in F_1$ such that, for each $\varepsilon > 0$, there exists $\delta^* \in (0, 1)$ such that, for every $\delta \geq \delta^*$, there exists $T \in \mathbb{N}$ such that, for every $g \in F_2$,*

$$P_{(f_\delta,g),\infty}\left(\left\{(x_1, x_2, \ldots) \in X^\infty : d\left(\frac{1-\delta}{1-\delta^t}\sum_{k=1}^{t} \delta^{k-1} x_k, S\right) \geq \varepsilon \text{ for some } t \in \{T, \ldots, \infty\}\right\}\right) < \varepsilon.$$

The above notion of approachability with discounting is analogous to the one given in Blackwell (1956), with $\frac{1-\delta}{1-\delta^t}\sum_{k=1}^{t} \delta^{k-1} x_k$ in place of $\frac{1}{t}\sum_{k=1}^{t} x_k$. Thus, while Blackwell's (1956) notion consider roughly the case $\delta = 1$, ours covers the case of δ sufficiently close but different than 1. As Blackwell's (1956) approachability, in the above definition, while T depends on both δ and ε, f_δ depends only on δ and not on ε. Also note that approachability with discounting concerns not just sufficiently large finite discounted averages of signals (when $t \geq T$ is finite) but also infinity discounted averages of signals (when $t = \infty$).

Note that if $S \subseteq \mathrm{R}^N$ is approachable then there exists $f \in F_1$ such that, for each $\varepsilon > 0$, there exists $T \in \mathbb{N}$ such that, for each $t \geq T$, there exists $\delta^*(t) \in (0,1)$ with

$$P_{(f,g),t}\left(\left\{d\left(\frac{1-\delta}{1-\delta^t}\sum_{k=1}^{t}\delta^{k-1}x_k, S\right) \geq \varepsilon\right\}\right) < \varepsilon$$

for each $\delta \geq \delta^*(t)$ and $g \in F_2$. This conclusion follows easily because, for each $t \in \mathbb{N}$, $\lim_{\delta\to 1}\frac{1-\delta}{1-\delta^t}\sum_{k=1}^{t}\delta^{k-1}x_k = \frac{1}{t}\sum_{k=1}^{t}x_k$ uniformly on $(x_1,\ldots,x_t) \in X^t$. But the requirement of our notion of approachability with discounting is to find some sufficiently large $\delta^* \in (0,1)$ such that, for each $\delta \geq \delta^*$, $\frac{1-\delta}{1-\delta^t}\sum_{k=1}^{t}\delta^{k-1}x_k$ is close to S with high probability uniformly on $g \in F_2$ and on all sufficiently large t. But this does not follow immediately from the definition of approachability.[6]

In some applications, our notion of approachability with discounting case may not be so user friendly as the minimum length of time T depends on the value of δ. For example to establish Folk theorems for repeated games, one typically requires a player to be able to guarantee that, with a sufficiently high probability, the discounted average of realized signals be close to a given set when the discount factor δ is sufficiently high and the discounted average is for some sufficiently long length $t(\delta)$ periods (e.g. Hörner and Lovo (2009)). The above notion of approachability however is for all $t \geq T$. Hence, it is not possible to apply our definition of approachability with discounting to such set-ups unless the length of time $t(\delta)$ required for the Folk theorem to be at least the length of time T (which also depends on δ) allowed by approachability with discounting.

The following notion of (δ, t)-approachability sidesteps this issue by focusing on a finite horizon t and by considering a bound on the probability of the discounted average of realized signals being within ε of a given set that depends explicitly on δ

[6] Approachability implies our notion of approachability with discounting if the following claim were to hold: For each $\varepsilon > 0$, there exists $\delta^* \in (0,1)$ such that, for each $\delta \geq \delta^*$, there exists $T \in \mathbb{N}$ such that, for each $(x_1, x_2, \ldots) \in X^\infty$ and $t \geq T$,

$$\left\|\frac{1-\delta}{1-\delta^t}\sum_{k=1}^{t}\delta^{k-1}x_k - \frac{1}{t}\sum_{k=1}^{t}x_k\right\| < \varepsilon.$$

But we cannot guarantee this.

and t. For each $\delta \in (0,1)$ and $t \in \mathbb{N}$, let

$$b(\delta,t) = \frac{1-\delta}{1+\delta}\frac{1+\delta^t}{1-\delta^t}.$$

Definition 3 ((δ,t)-approachability) *Given $\delta \in (0,1)$ and $t \in \mathbb{N}$, a subset S of $\mathbb{R}^N$ is* (δ,t)-approachable *if there exists $f \in F_1$ such that, for each $\varepsilon > 0$ and $g \in F_2$,*

$$P_{(f,g),t}\left(\left\{(x_1,\ldots,x_t) \in X^t : d\left(\frac{1-\delta}{1-\delta^t}\sum_{k=1}^{t}\delta^{k-1}x_k, S\right) \geq \varepsilon\right\}\right) < \left(\frac{M}{\varepsilon} - \varepsilon\right) ce^{-\frac{d\varepsilon^4}{b(\delta,t)}} + \varepsilon$$

where $c, d, M > 0$ depend only on X and S.

Simple calculation shows that $\lim_{(\delta,t)\to(1,\infty)} b(\delta,t) = 0$ (see Lemma 2 in the Appendix). Hence, if a set $S \subseteq \mathbb{R}^N$ is (δ,t)-approachable for each $\delta \in (0,1)$ and $t \in \mathbb{N}$, then for each $\varepsilon > 0$, there exists $\delta^* \in (0,1)$ and $T \in \mathbb{N}$ such that, for every $\delta \geq \delta^*$, $t \geq T$ there exists $f_{\delta,t} \in F_1$ such that for all $g \in F_2$,

$$P_{(f_{\delta,t},g),t}\left(\left\{(x_1,\ldots,x_t) \in X^t : d\left(\frac{1-\delta}{1-\delta^t}\sum_{k=1}^{t}\delta^{k-1}x_k, S\right) \geq \varepsilon\right\}\right) < \varepsilon.$$

Note that in some important cases (e.g. Theorem 3 below) the strategy $f_{\delta,t}$ in the above expression can be made independent of δ and t.

Our main result characterizes the convex subsets of $\mathbb{R}^N$ that are approachable. This characterization uses the following notion.

Definition 4 (Securability) *A subset S of $\mathbb{R}^N$ is* securable *if, there exists $c, d > 0$ such that, for each $g \in F_2$, there exists $f \in F_1$ such that, for each $\varepsilon > 0$, $t \in \mathbb{N}$ and $\delta \in (0,1)$,*

$$P_{(f,g),t}\left(\left\{(x_1,\ldots,x_t) \in X^t : d\left(\frac{1-\delta}{1-\delta^t}\sum_{k=1}^{t}\delta^{k-1}x_k, S\right) \geq \varepsilon\right\}\right) < ce^{-\frac{d\varepsilon^2}{b(\delta,t)}}.$$

The usefulness of securability is seen in the following result that shows that it implies approachability with discounting and (δ,t)-approachability for each $\delta \in (0,1)$ and $t \in \mathbb{N}$.

Theorem 1 *If $S \subseteq \mathbb{R}^N$ is securable, then S is approachable with discounting and (δ,t)-approachable for each $\delta \in (0,1)$ and $t \in \mathbb{N}$.*

The proof of Theorem 1 as well as the other ones in this section rely on the following concentration result that may have interest in its own right.

Lemma 1 *For any* $(f,g) \in F$, $\varepsilon > 0$, $\delta \in (0,1)$ *and* $t \in \mathbb{N}$,

$$P_{(f,g),t}\left(S_{\delta,t,\varepsilon}\right) \leq |X| e^{-\frac{2\varepsilon^2}{B^2 b(\delta,t)}},$$

where

$$S_{\delta,t,\varepsilon} = \left\{ (x_1,\ldots,x_t) \in X^t : \left\| \frac{1-\delta}{1-\delta^t} \sum_{k=1}^{t} \delta^{k-1} x_k - \frac{1-\delta}{1-\delta^t} \sum_{k=1}^{t} \delta^{k-1} \sum_{x\in X} \beta_x(x_1,\ldots,x_{k-1}) x \right\| \geq \varepsilon \right\}$$

and $B = |X| \max_{x\in X} ||x||$.

Since $\lim_{(\delta,t)\to(1,\infty)} b(\delta,t) = 0$, Lemma 1 shows that the discounted sum of the realization of signals is close to its expected discounted sum with a probability close to one when δ is close to one and t is close to infinity. In other words, the distribution of the discounted sum of signals is concentrated around its expected value.

Theorem 2 below characterizes the convex subset of an Euclidean space that are approachable with discounting. In addition, it shows that all the notions we considered in this paper — approachability, approachability with discounting, (δ,t)-approachability for each δ and t, and securability — are equivalent.

It uses the following notation: For each $i \in A_1$ and $j \in A_2$, let $\bar{m}(i,j) = \sum_{x\in X} x m_x(i,j)$ be the expected value of x with respect to $m(i,j)$ and, for each $q \in Q$, let

$$T(q) = \mathrm{co}\left\{ \sum_{j=1}^{s} q_j \bar{m}(1,j), \ldots, \sum_{j=1}^{s} q_j \bar{m}(r,j) \right\}.$$

Theorem 2 *Let* $S \subseteq \mathrm{R}^N$ *be closed and convex. Then the following are equivalent:*

1. $S \cap T(q) \neq \emptyset$ *for each* $q \in Q$.

2. S *is approachable.*

3. S *is approachable with discounting.*

4. S *is* (δ,t)*-approachable for each* $\delta \in (0,1)$ *and* $t \in \mathbb{N}$.

5. S is securable.

For each $p \in P$, let

$$R(p) = \text{co}\left\{\sum_{i=1}^{r} p_i \bar{m}(i,1), \ldots, \sum_{i=1}^{r} p_i \bar{m}(i,s)\right\}.$$

We have that $R(p) \cap T(q) \neq \emptyset$; hence, $R(p)$ is approachable with discounting. In this case, we obtain a stronger conclusion, namely that player 1's strategy can be taken to be identically equal to p and, in particular, independent of δ.

Theorem 3 *If $R(p) \subseteq S$ for some $p \in P$ and $f \equiv p$, then, for every $\varepsilon > 0$, there exists $\delta^* \in (0,1)$ and $T^* \in \mathbb{N}$ such that, for every $\delta \geq \delta^*$, $t \geq T^*$ and $g \in F_2$,*

$$P_{(f,g),t}\left(\left\{(x_1, \ldots, x_t) \in X^t : d\left(\frac{1-\delta}{1-\delta^t}\sum_{k=1}^{t} \delta^{k-1} x_k, S\right) \geq \varepsilon\right\}\right) < \varepsilon.$$

In addition, for every $\delta \geq \delta^*$, there exists $T \in \mathbb{N}$ such that, for every $g \in F_2$,

$$P_{(f,g),\infty}\left(\left\{(x_1, x_2, \ldots) \in X^\infty : d\left(\frac{1-\delta}{1-\delta^t}\sum_{k=1}^{t} \delta^{k-1} x_k, S\right) \geq \varepsilon \text{ for some } t \in \{T, \ldots, \infty\}\right\}\right) < \varepsilon.$$

Theorem 3 is analogous to Corollary 1 in Blackwell (1956). As in the latter, $R(p)$, and therefore any superset of it, is approachable with discounting for each $p \in P$ using the constant strategy $f \equiv p$. However, we note that Theorem 3 does not follow from Blackwell's (1956) Corollary 1 for the same reason why approachability with discounting does not follows from approachability.

We also note here that our approach uses the minmax theorem for two-person, zero-sum games which is applied to the game where player 1 seeks to minimize the expected distance of the infinitely discounted sum of realized public signals to a given set S using a repeated game strategy. Despite this latter feature, the game is a one-shot game. Consequently, our approach is not based on Zamir's (1992) results for repeated two-person, zero-sum games.

A Appendix

A.1 Lemmas

In this section, we state and prove some useful lemmas. We start with the proof of Lemma 1, which uses a concentration result from probability theory.

Proof of Lemma 1. Let $(f, g) \in F$, $\varepsilon > 0$, $\delta \in (0, 1)$ and $t \in \mathbb{N}$ be given. For each $x \in X$, define

$$F_x(x_1, \ldots, x_t) = \frac{1-\delta}{1-\delta^t} \sum_{k=1}^{t} \delta^{k-1} \left(1_x(x_k) - \beta_x(x_1, \ldots, x_{k-1})\right)$$

for each $(x_1, \ldots, x_t) \in X^t$. We first argue that it suffices to show that

$$P_{(f,g),t}\left(|F_x(x_1, \ldots, x_t)| \geq \frac{\varepsilon}{B}\right) \leq e^{-\frac{2\varepsilon^2}{B^2 b(\delta,t)}} \text{ for each } x \in X. \tag{2}$$

Indeed, we have that

$$\left\| \frac{1-\delta}{1-\delta^t} \sum_{k=1}^{t} \delta^{k-1} x_k - \frac{1-\delta}{1-\delta^t} \sum_{k=1}^{t} \delta^{k-1} \sum_{x \in X} \beta_x(x_1, \ldots, x_{k-1}) x \right\| =$$
$$\left\| \frac{1-\delta}{1-\delta^t} \sum_{k=1}^{t} \delta^{k-1} \sum_{x \in X} x 1_x(x_k) - \frac{1-\delta}{1-\delta^t} \sum_{k=1}^{t} \delta^{k-1} \sum_{x \in X} \beta_x(x_1, \ldots, x_{k-1}) x \right\| =$$
$$\left\| \sum_{x \in X} x \left(\frac{1-\delta}{1-\delta^t} \sum_{k=1}^{t} \delta^{k-1} 1_x(x_k) - \frac{1-\delta}{1-\delta^t} \sum_{k=1}^{t} \delta^{k-1} \beta_x(x_1, \ldots, x_{k-1}) \right) \right\| =$$
$$\left\| \sum_{x \in X} x F_x(x_1, \ldots, x_t) \right\| \leq \sum_{x \in X} ||x|| \, |F_x(x_1, \ldots, x_t)| \, .$$

Hence,

$$\left\{ (x_1, \ldots, x_t) \in X^t : \left\| \frac{1-\delta}{1-\delta^t} \sum_{k=1}^{t} \delta^{k-1} x_k - \frac{1-\delta}{1-\delta^t} \sum_{k=1}^{t} \delta^{k-1} \sum_{x \in X} \beta_x(x_1, \ldots, x_{k-1}) x \right\| \geq \varepsilon \right\}$$
$$\subseteq \bigcup_{x \in X} \left\{ (x_1, \ldots, x_t) \in X^t : |F_x(x_1, \ldots, x_t)| \geq \frac{\varepsilon}{B} \right\}$$

and, therefore, if (2) holds, then

$$P_{(f,g),t}\left(\left\| \frac{1-\delta}{1-\delta^t} \sum_{k=1}^{t} \delta^{k-1} x_k - \frac{1-\delta}{1-\delta^t} \sum_{k=1}^{t} \delta^{k-1} \sum_{x \in X} \beta_x(x_1, \ldots, x_{k-1}) x \right\| \geq \varepsilon \right) \leq |X| e^{-\frac{2\varepsilon^2}{B^2 b(\delta,t)}}.$$

By the above, in the remaining of this proof, we establish (2). Fix $x \in X$. For convenience, for each $1 \leq k \leq t$, we write P_k (resp. π_k) instead of $P_{(f,g),k}$ (resp. $\pi_{(f,g),k}$). First, note that

$$\begin{aligned} E(F_x) &= \frac{1-\delta}{1-\delta^t}\sum_{k=1}^{t}\delta^{k-1}\left(\pi_k(x) - \sum_{(x_1,\ldots,x_{k-1})\in X^{k-1}} P_k(x_1,\ldots,x_{k-1})\beta_x(x_1,\ldots,x_{k-1})\right) \\ &= \frac{1-\delta}{1-\delta^t}\sum_{k=1}^{t}(\pi_k(x)-\pi_k(x)) = 0. \end{aligned}$$

Next, fix $k \in \{1,\ldots,t\}$ and $\hat{x}_1,\ldots,\hat{x}_{k-1} \in X^{k-1}$. Let $B_k = \{(x_1,\ldots,x_t) \in X^t : x_i = \hat{x}_i \text{ for all } i = 1,\ldots,k-1\}$ and, for each $x' \in X$,

$$g_k(x') = E(F_x(x_1,\ldots,x_t)|B_k, x_k = x') - E(F_x(x_1,\ldots,x_t)|B_k).$$

Furthermore, let $\mathrm{ran}(\hat{x}_1,\ldots,\hat{x}_{k-1}) = \sup\{|g_k(x') - g_k(\bar{x})| : x',\bar{x} \in X\}$. We have that, for each $x' \in X$ and $l \in \{k+1,\ldots,t\}$,

$$E(1_x(x_l) - \beta_x(x_1,\ldots,x_{l-1})|B_k, x_k = x') = 0 \tag{3}$$

since

$$\begin{aligned} &E(1_x(x_l)|B_k, x_k = x') = \\ &\sum_{(x_{k+1},\ldots,x_{l-1})}\prod_{n=k+1}^{l-1}\beta_{x_n}(\hat{x}_1,\ldots,\hat{x}_{k-1},x',\ldots,x_{n-1})\beta_x(\hat{x}_1,\ldots,\hat{x}_{k-1},x',x_{k+1},\ldots,x_{l-1}) = \\ &E(\beta_x(x_1,\ldots,x_{l-1})|B_k, x_k = x'). \end{aligned}$$

Hence, it follows from (3) that

$$|g_k(x') - g_k(\bar{x})| = \left|\frac{1-\delta}{1-\delta^t}\delta^{k-1}\left(1_x(x') - 1_x(\bar{x})\right)\right|$$

and, hence,

$$\mathrm{ran}(\hat{x}_1,\ldots,\hat{x}_{k-1}) = \frac{1-\delta}{1-\delta^t}\delta^{k-1}.$$

For each $(\hat{x}_1,\ldots,\hat{x}_t) \in X^t$, let $R^2(\hat{x}_1,\ldots,\hat{x}_t) = \sum_{k=1}^{t}(\mathrm{ran}(\hat{x}_1,\ldots,\hat{x}_{k-1}))^2$ and

$$\hat{r}^2 = \sup_{(\hat{x}_1,\ldots,\hat{x}_t)\in X^t} R^2(\hat{x}_1,\ldots,\hat{x}_t).$$

Thus, we obtain that

$$R^2(\hat{x}_1, \ldots, \hat{x}_t) = \left(\frac{1-\delta}{1-\delta^t}\right)^2 \sum_{k=1}^{t} \delta^{2(k-1)}$$

and, hence,

$$\hat{r}^2 = \left(\frac{1-\delta}{1-\delta^t}\right)^2 \sum_{k=1}^{t} \delta^{2(k-1)} = \frac{(1-\delta)^2}{(1-\delta^t)^2} \frac{1-\delta^{2t}}{1-\delta^2} = \frac{1-\delta}{1+\delta} \frac{1+\delta^t}{1-\delta^t} = b(\delta, t).$$

It follows by Theorem 3.7 in McDiarmid (1998) that

$$P_{(f,g),t}\left(|F_x(x_1, \ldots, x_t)| \geq \frac{\varepsilon}{B}\right) \leq e^{-\frac{2\varepsilon^2}{B^2 b(\delta,t)}},$$

as desired. This completes the proof. ■

Lemma 2 $\lim_{(\delta,t)\to(1,\infty)} b(\delta, t) = 0.$

Proof. Let $\{(\delta_n, t_n)\}_{n=1}^{\infty}$ be a sequence in $(0,1) \times \mathbb{N}$ such that $(\delta_n, t_n) \to (1, \infty)$. As $0 \leq \delta_n^{t_n} \leq 1$ for each $n \in \mathbb{N}$, taking a subsequence if necessary, we may assume that $\{\delta_n^{t_n}\}_{n=1}^{\infty}$ converges; let $\alpha = \lim_n \delta_n^{t_n} \in [0,1]$. When $\alpha \neq 1$, it is clear that $\lim_n b(\delta_n, t_n) = 0$. Thus, assume that $\alpha = 1$.

Let $\varepsilon > 0$ and pick $T \in \mathbb{N}$ such that $1/T < \varepsilon$. Thus, for each n sufficiently large, $t_n > T$ and, therefore,

$$0 \leq \frac{1-\delta_n}{1-\delta_n^{t_n}} \leq \frac{1-\delta_n}{1-\delta_n^T} \to \frac{1}{T} < \varepsilon.$$

Thus, $\lim_n \frac{1-\delta_n}{1-\delta_n^{t_n}} = 0$ and, because $(1+\delta_n^{t_n})/(1+\delta_n) \to 1$, it follows that $\lim_n b(\delta_n, t_n) = 0$. ■

Corollary 1 *For each $\gamma > 0$, there exists $\delta_\gamma \in (0,1)$ and $T_\gamma \in \mathbb{N}$ such that $\delta \geq \delta_\gamma$ and $t \geq T_\gamma$ imply that*

$$|X| e^{-\frac{2\varepsilon^2}{B^2 b(\delta,t)}} < \gamma.$$

Proof. Indeed, if $\eta > 0$ is such that $b(\delta, t) < \eta$ implies that $|X| e^{-\frac{2\varepsilon^2}{B^2 b(\delta,t)}} < \gamma$, then use Lemma 2 to obtain δ_γ and T_γ. ■

It follows by Lemmas 1 and 2 that we can focus on the expected discounted sum of signals $\frac{1-\delta}{1-\delta^t} \sum_{k=1}^{t} \delta^{k-1} \sum_{x \in X} \beta_x(x_1, \ldots, x_{k-1}) x$ instead of the discounted sum of the realized signals $\frac{1-\delta}{1-\delta^t} \sum_{k=1}^{t} \delta^{k-1} x_k$. That this is convenient is shown in the following three lemmas.

Lemma 3 *If $f \equiv p$, then*

$$\frac{1-\delta}{1-\delta^t}\sum_{k=1}^{t}\delta^{k-1}\sum_{x\in X}\beta_x(x_1,\ldots,x_{k-1})x \in R(p)$$

for all $t \in \mathbb{N}$, $(x_1,\ldots,x_t) \in X^t$, $\delta \in (0,1)$ and $g \in F_2$.

Proof. Let $t \in \mathbb{N}$, $(x_1,\ldots,x_t) \in X^t$, $\delta \in (0,1)$ and $g \in F_2$ be given. The definition of f implies that, for each $1 \leq k \leq t$,

$$\sum_{x\in X}\beta_x(x_1,\ldots,x_{k-1})x = \sum_{j=1}^{s} g_j(x_1,\ldots,x_{k-1})\sum_{i=1}^{r} p_i\bar{m}(i,j) \in R(p).$$

Hence, as $R(p)$ is convex and $\frac{1-\delta}{1-\delta^t}\sum_{k=1}^{t}\delta^{k-1} = 1$,

$$\frac{1-\delta}{1-\delta^t}\sum_{k=1}^{t}\delta^{k-1}\sum_{x\in X}\beta_x(x_1,\ldots,x_{k-1})x \in R(p).$$

This concludes the proof. ■

The following lemma is analogous to Lemma 3.

Lemma 4 *If $g \equiv q$, then*

$$\frac{1-\delta}{1-\delta^t}\sum_{k=1}^{t}\delta^{k-1}\sum_{x\in X}\beta_x(x_1,\ldots,x_{k-1})x \in T(q)$$

for all $t \in \mathbb{N}$, $(x_1,\ldots,x_t) \in X^t$, $\delta \in (0,1)$ and $f \in F_1$.

Lemma 5 *Let $S \subseteq \mathrm{R}^N$ be convex and such that $S \cap T(q) \neq \emptyset$ for each $q \in Q$. Then, for each $g \in F_2$, there exists $f \in F_1$ such that*

$$\frac{1-\delta}{1-\delta^t}\sum_{k=1}^{t}\delta^{k-1}\sum_{x\in X}\beta_x(x_1,\ldots,x_{k-1})x \in S$$

for all $t \in \mathbb{N}$, $(x_1,\ldots,x_t) \in X^t$ and $\delta \in (0,1)$.

Proof. Let $g \in F_2$ be given. For each $t \in \mathbb{N}$ and $(x_1,\ldots,x_t) \in X^t$, there exists $z \in S \cap T(g(x_1,\ldots,x_t))$. Hence, for some $p \in P$,

$$z = \sum_{i=1}^{r} p_i \sum_{j=1}^{s} g_j(x_1,\ldots,x_t)\bar{m}(i,j).$$

Define $f(x_1, \ldots, x_t) = p$.

Fix $t \in \mathbb{N}$, $(x_1, \ldots, x_t) \in X^t$ and $\delta \in (0,1)$. The definition of f implies that, for each $1 \leq k \leq t$,

$$\sum_{x \in X} \beta_x(x_1, \ldots, x_{k-1})x = \sum_{i=1}^{r} f_i(x_1, \ldots, x_{k-1}) \sum_{j=1}^{s} g_j(x_1, \ldots, x_{k-1})\bar{m}(i,j) \in S.$$

Hence, as S is convex and $\frac{1-\delta}{1-\delta^t} \sum_{k=1}^{t} \delta^{k-1} = 1$,

$$\frac{1-\delta}{1-\delta^t} \sum_{k=1}^{t} \delta^{k-1} \sum_{x \in X} \beta_x(x_1, \ldots, x_{k-1})x \in S.$$

This concludes the proof. ■

The following simple lemma is used in the proof of our results.

Lemma 6 *For each $\varepsilon > 0$ and $\delta \in (0,1)$, there exits $T \in \mathbb{N}$ such that, for each $t \geq T$ and $(x_1, \ldots, x_t) \in X^t$,*

$$\left\| \frac{1-\delta}{1-\delta^t} \sum_{k=1}^{t} \delta^{k-1} x_k - (1-\delta) \sum_{k=1}^{\infty} \delta^{k-1} x_k \right\| < \varepsilon.$$

Proof. Let $B = \max_{x \in X} ||x||$, and pick T such that $2B\delta^T < \varepsilon$. We then have that, for each $t \geq T$,

$$\left\| \frac{1-\delta}{1-\delta^t} \sum_{k=1}^{t} \delta^{k-1} x_k - (1-\delta) \sum_{k=1}^{\infty} \delta^{k-1} x_k \right\| \leq$$
$$B(1-\delta) \left| \left(1 - \frac{1}{1-\delta^t}\right) \sum_{k=1}^{t} \delta^{k-1} \right| + B(1-\delta) \sum_{k=t+1}^{\infty} \delta^{k-1} =$$
$$B(1-\delta) \frac{\delta^t}{1-\delta^t} \frac{1-\delta^t}{1-\delta} + B\delta^t = 2B\delta^t < \varepsilon.$$

■

A.2 Proof of Theorem 1

First, some additional notation. For each $\delta \in (0,1)$, $t \in \mathbb{N}$ and $\varepsilon > 0$, let

$$A_{\delta,t,\varepsilon} = \left\{ (x_1,\ldots,x_t) \in X^t : d\left(\frac{1-\delta}{1-\delta^t} \sum_{k=1}^{t} \delta^{k-1} x_k, S \right) \geq \varepsilon \right\},$$

$$\tilde{A}_{\delta,t,\varepsilon} = \left\{ (x_1,x_2,\ldots) \in X^\infty : d\left(\frac{1-\delta}{1-\delta^t} \sum_{k=1}^{t} \delta^{k-1} x_k, S \right) \geq \varepsilon \right\} \text{ and}$$

$$A_{\delta,\varepsilon} = \left\{ (x_1,x_2,\ldots) \in X^\infty : d\left((1-\delta) \sum_{k=1}^{\infty} \delta^{k-1} x_k, S \right) \geq \varepsilon \right\}.$$

A.2.1 Securability implies approachability with discounting

Let $S \subseteq \mathbb{R}^N$ be securable. We show that S is approachable with discounting in several steps.

(I) *S being securable implies that, for each $g \in F_2$, there exists $f \in F_1$ such that, for each $\varepsilon > 0$ and $\delta \in (0,1)$,*

$$P_{(f,g),\infty}(A_{\delta,\varepsilon}) < ce^{-\frac{d'\varepsilon^2}{1-\delta}}$$

where $d' = d(1+\delta)/8$.

Indeed, fix $g \in F_2$ and let $f \in F_1$ be given by the definition of securability. Let $\varepsilon > 0$ and $\delta \in (0,1)$ be given. Then

$$P_{(f,g),t}\left(A_{\delta,t,\varepsilon/2}\right) < ce^{-\frac{d\varepsilon^2}{4b(\delta,t)}}.$$

Let T be given by Lemma 6 with $\varepsilon/2$ in place of ε and $T' > T$ be such that

$$ce^{-\frac{d\varepsilon^2}{4b(\delta,T')}} < ce^{-\frac{d\varepsilon^2(1+\delta)}{8(1-\delta)}};$$

such T' exists since $\lim_{t\to\infty} b(\delta,t) = (1-\delta)/(1+\delta)$ and $ce^{-\frac{d\varepsilon^2(1+\delta)}{4(1-\delta)}} < ce^{-\frac{d\varepsilon^2(1+\delta)}{8(1-\delta)}}$. It then follows that $A_{\delta,\varepsilon} \subseteq \tilde{A}_{\delta,T',\varepsilon/2}$ and, therefore,

$$P_{(f,g),\infty}(A_{\delta,\varepsilon}) \leq P_{(f,g),\infty}(\tilde{A}_{\delta,T',\varepsilon/2}) = P_{(f,g),T'}(A_{\delta,T',\varepsilon/2}) < ce^{-\frac{d\varepsilon^2}{4b(\delta,T')}} < ce^{-\frac{d\varepsilon^2(1+\delta)}{8(1-\delta)}}.$$

(II) *For each $\delta \in (0,1)$, the function $v_\delta : X^\infty \to \mathbb{R}$ defined by setting, for each $x^\infty \in X^\infty$,*

$$v_\delta(x^\infty) = d((1-\delta)\sum_{k=1}^{\infty} \delta^{k-1} x_k, S)$$

is continuous and bounded.

Let $M > 1$ be such that $v_\delta(x^\infty) \leq M$ for each $\delta \in (0,1)$ and $x^\infty \in X^\infty$. That such M exists follows because $v_\delta(x^\infty) \leq \max_{x \in \mathrm{co}(X)} d(x,S)$ for each $\delta \in (0,1)$ and $x^\infty \in X^\infty$; as $\mathrm{co}(X)$ is compact and the function $x \mapsto d(x,S)$, mapping $\mathrm{co}(X)$ into $\mathbb{R}$ is continuous and bounded, then $\max_{x \in \mathrm{co}(X)} d(x,S) < \infty$.

We note here for later use that v_δ is continuous since the function $x^\infty \mapsto (1-\delta)\sum_{k=1}^{\infty} \delta^{k-1} x_k$, from X^∞ to $\mathrm{co}(X)$, is continuous and, as noted already, so is the function $x \mapsto d(x,S)$.

(III) *S being securable implies that:*

$$\begin{aligned}&\textit{For each } g \in F_2, \textit{ there exists } f \in F_1 \textit{ such that, for each } \varepsilon > 0 \textit{ and } \delta \in (0,1),\\ &\int_{X^\infty} v_\delta \mathrm{d}P_{(f,g),\infty} < (M-\varepsilon)ce^{-\frac{d'\varepsilon^2}{1-\delta}} + \varepsilon.\end{aligned} \tag{4}$$

Indeed, fix $g \in F_2$ and let $f \in F_1$ be given by (I). Let $\varepsilon > 0$ and $\delta \in (0,1)$ be given. Noting that $\{v_\delta \geq \varepsilon\} = A_{\delta,t,\varepsilon}$, it follows that

$$\int_{X^\infty} v_\delta \mathrm{d}P_{(f,g),\infty} < Mce^{-\frac{d'\varepsilon^2}{1-\delta}} + \varepsilon(1 - ce^{-\frac{d'\varepsilon^2}{1-\delta}}).$$

(IV) *S is approachable with discounting if the following condition holds:*

$$\begin{aligned}&\textit{For each } \delta \in (0,1), \textit{there exists } f \in F_1 \textit{ such that, for each } \varepsilon > 0, \textit{ there exists}\\ &\delta^* \in (0,1) \textit{ such that, for each } \delta \geq \delta^* \textit{ and } g \in F_2, P_{(f,g),\infty}(A_{\delta,\varepsilon}) < \varepsilon.\end{aligned} \tag{5}$$

Indeed, for each $\delta \in (0,1)$, let $f \in F_1$ be given by (5). Let $\varepsilon > 0$ be given and let $\delta^* \in (0,1)$ be obtained from condition (5) corresponding to $\varepsilon/2$. Let $\delta \geq \delta^*$ and let $T \in \mathbb{N}$ be given by Lemma 6 corresponding to δ and $\varepsilon/2$. We then have that $(\bigcup_{t=T}^{\infty} \tilde{A}_{\delta,t,\varepsilon}) \cup A_{\delta,\varepsilon} \subseteq A_{\delta,\varepsilon/2}$, implying that, for each $g \in F_2$,

$$P_{(f,g),\infty}((\bigcup_{t=T}^{\infty} \tilde{A}_{\delta,t,\varepsilon}) \cup A_{\delta,\varepsilon}) \leq P_{(f,g),\infty}(A_{\delta,\varepsilon/2}) < \varepsilon/2 < \varepsilon.$$

(V) *S is approachable with discounting if the following condition holds:*

$$\begin{aligned}&\textit{For each } \delta \in (0,1), \textit{there exists } f \in F_1 \textit{ such that, for each } \varepsilon > 0, \textit{ there exists}\\ &\delta^* \in (0,1) \textit{ such that, for each } \delta \geq \delta^* \textit{ and } g \in F_2, \int_{X^\infty} v_\delta \mathrm{d}P_{(f,g),\infty} < \varepsilon.\end{aligned} \tag{6}$$

This step holds because we can show that condition (6) implies (5): For each $\delta \in (0,1)$, let f be given by (6). Let $\varepsilon > 0$ be given and $\delta^* \in (0,1)$ be obtained from (6) corresponding to ε^2. Thus, for each $\delta \geq \delta^*$ and $g \in F_2$,

$$P_{(f,g),\infty}\left(A_{\delta,\varepsilon}\right) = P_{(f,g),\infty}\left(\{v_\delta \geq \varepsilon\}\right) \leq \frac{\int_{X^\infty} v_\delta \mathrm{d}P_{(f,g),\infty}}{\varepsilon} < \frac{\varepsilon^2}{\varepsilon} = \varepsilon.$$

(VI) *Condition* (4) *implies condition* (6).

Fix $\delta \in (0,1)$ and consider the following two-person zero-sum game G: Players are $N = \{1,2\}$ and player i's strategy space is the set Σ_i of Borel probability measures over pure strategies $f_i : H \to A_i$; since H is countable and A_i is finite, the set of pure strategies of player i, A_i^H endowed with the product topology, is a compact metric space. As for payoffs: Letting $F_p \subseteq F$ denote the set of pure strategy profiles, player 1's payoff function is $\hat{u} : F_p \to \mathbb{R}$ defined by setting, for each $(f,g) \in F_p$,

$$\hat{u}(f,g) = -\int_{X^\infty} v_\delta \mathrm{d}P_{(f,g),\infty};$$

$u : \Sigma \to \mathbb{R}$ is the mixed extension of $\hat{u}$: $u(\sigma_1,\sigma_2) = \int_{F_p} \hat{u}\mathrm{d}(\sigma_1 \times \sigma_2)$ for each $(\sigma_1,\sigma_2) \in \Sigma = \Sigma_1 \times \Sigma_2$.

We have that $\hat{u}$ is continuous. Fix $(f,g) \in F_p$ and $(f_k,g_k) \to (f,g)$. Note first that X^∞ is a compact metrizable space when endowed with the product topology. Letting $M(X^\infty)$ denote the set of Borel probability measures on X^∞, we have that $M(X^\infty)$ is a compact metric space when endowed with the narrow topology.

We have that $P_{(f_k,g_k),\infty} \to P_{(f,g),\infty}$. Since $M(X^\infty)$ is a compact metric space, we may assume that the sequence $\{P_{(f_k,g_k),\infty}\}_{k=1}^\infty$ converges; let $\Psi = \lim_k P_{(f_k,g_k),\infty}$. Fix $t \in \mathbb{N}$ and $B \subseteq X^t$. As X^t is finite, the function $1_{B\times X\times X\times\cdots} : X^\infty \to \{0,1\}$ is continuous. Thus,

$$\begin{aligned}&\Psi(B \times X \times X \times \cdots) = \int_{X^\infty} 1_{B\times X\times X\times\cdots}\mathrm{d}\Psi = \lim_k \int_{X^\infty} 1_{B\times X\times X\times\cdots}\mathrm{d}P_{(f_k,g_k),\infty} = \\ &\lim_k P_{(f_k,g_k),\infty}(B \times X \times X \times \cdots) = \lim_k P_{(f_k,g_k),t}(B) = P_{(f,g),t}(B) = \\ &P_{(f,g),\infty}(B \times X \times X \times \cdots),\end{aligned}$$

from which $\Psi = P_{(f,g),\infty}$ follows. The continuity of $\hat{u}$ now follows since v_δ is continuous and bounded by (II); indeed,

$$\hat{u}(f_k,g_k) = \int_{X^\infty} v_\delta \mathrm{d}P_{(f_k,g_k),\infty} \to \int_{X^\infty} v_\delta \mathrm{d}P_{(f,g),\infty} = \hat{u}(f,g).$$

The Minmax Theorem (see Mertens (1986, p. 237)) implies that

$$\max_{\sigma_1\in\Sigma_1}\min_{\sigma_2\in\Sigma_2} u(\sigma_1,\sigma_2) = \min_{\sigma_2\in\Sigma_2}\max_{\sigma_1\in\Sigma_1} u(\sigma_1,\sigma_2).$$

Kuhn's Theorem (for infinite extensive-form games as in Aumann (1964) or for finite extensive-form games as in Kuhn (1953) and Selten (1975), the latter together with Lemma 6) implies that

$$\max_{f\in F_1}\min_{g\in F_2} u(f,g) = \min_{g\in F_2}\max_{f\in F_1} u(f,g).$$

Hence, let $f_\delta \in F_1$ be such that $\min_{g\in F_2} u(f_\delta,g) = \max_{f\in F_1}\min_{g\in F_2} u(f,g)$.

Let $\varepsilon > 0$ be given. Now if (4) holds, then

$$\min_{g\in F_2}\max_{f\in F_1} u(f,g) > -\left[\left(M-\frac{\varepsilon}{2}\right) ce^{-\frac{d'\varepsilon^2}{4(1-\delta)}} + \frac{\varepsilon}{2}\right]$$

and, hence,

$$-\int_{X^\infty} v_\delta \mathrm{d}P_{(f_\delta,g),\infty} \geq \min_{g\in F_2} u(f_\delta,g) > -\left[\left(M-\frac{\varepsilon}{2}\right) ce^{-\frac{d'\varepsilon^2}{4(1-\delta)}} + \frac{\varepsilon}{2}\right].$$

Since $\lim_{\delta\to 1}\left[\left(M-\frac{\varepsilon}{2}\right) ce^{-\frac{d'\varepsilon^2}{4(1-\delta)}} + \frac{\varepsilon}{2}\right] = \frac{\varepsilon}{2} < \varepsilon$, there exists $\delta^* \in (0,1)$ such that, for each $\delta \geq \delta^*$ and $g \in F_2$,

$$\int_{X^\infty} v_\delta \mathrm{d}P_{(f^*,g),\infty} < \varepsilon.$$

This establishes the condition (6) and concludes the proof.

A.2.2 Securability implies (δ, t)-approachability for each δ and t

Let $S \subseteq \mathbb{R}^N$ be securable. We show that S is (t,δ)-approachable for each $\delta \in (0,1)$ and $t \in \mathbb{N}$ in several steps, analogous to the ones in Section A.2.1. Let $\delta \in (0,1)$ and $t \in \mathbb{N}$.

(I) *S being securable implies that, for each $g \in F_2$, there exists $f \in F_1$ such that, for each $\varepsilon > 0$, $P_{(f,g),t}(A_{\delta,t,\varepsilon}) < ce^{-\frac{d\varepsilon^2}{b(\delta,t)}}$.*

(II) *For each $\delta \in (0,1)$ and $t \in \mathbb{N}$, the function $v_{\delta,t} : X^t \to \mathbb{R}$ defined by setting, for each $x^t \in X^t$,*

$$v_{\delta,t}(x^t) = d\left(\frac{1-\delta}{1-\delta^t}\sum_{k=1}^{t}\delta^{k-1}x_k, S\right).$$

is continuous and bounded. This is clear since X^t is finite.

(III) *S being securable implies that:*

$$\text{For each } g \in F_2, \text{ there exists } f \in F_1 \text{ such that, for each } \varepsilon > 0, t \in \mathbb{N} \text{ and } \delta \in (0,1),$$
$$\int_{X^t} v_{\delta,t} \mathrm{d}P_{(f,g),t} < (M-\varepsilon)ce^{-\frac{d\varepsilon^2}{b(\delta,t)}} + \varepsilon. \tag{7}$$

(IV) *S is (δ, t)-approachable if the following condition holds:*

$$\text{There exists } f \in F_1 \text{ such that, for each } \varepsilon > 0 \text{ and } g \in F_2,$$
$$\int_{X^t} v_{\delta,t} \mathrm{d}P_{(f,g),t} < (M-\varepsilon)ce^{-\frac{d\varepsilon^2}{b(\delta,t)}} + \varepsilon. \tag{8}$$

Indeed, for each given $\varepsilon > 0$ and $g \in F_2$, using (8) with ε^2 in place of ε, it follows that

$$\begin{aligned} P_{(f,g),t}\left(A_{\delta,t,\varepsilon}\right) &= P_{(f,g),t}\left(\{v_{\delta,t} \geq \varepsilon\}\right) \leq \frac{\int_{X^t} v_{\delta,t} \mathrm{d}P_{(f,g),t}}{\varepsilon} \\ &< \frac{(M-\varepsilon^2)ce^{-\frac{d\varepsilon^4}{b(\delta,t)}} + \varepsilon^2}{\varepsilon} = \left(\frac{M}{\varepsilon} - \varepsilon\right) ce^{-\frac{d\varepsilon^4}{b(\delta,t)}} + \varepsilon. \end{aligned}$$

(V) *Condition* (7) *implies condition* (8)*.*

Fix $\delta \in (0,1)$ and consider the following two-person zero-sum game G: Players are $N = \{1,2\}$ and player i's strategy space is the set Σ_i of probability measures over pure strategies $f_i : \cup_{k=0}^{t-1} H_k \to A_i$; since X and A_i are finite, the set of pure strategies of player i is also finite. As for payoffs: Letting $F_p \subseteq F$ denote the set of pure strategy profiles, player 1's payoff function is $\hat{u} : F_p \to \mathbb{R}$ defined by setting, for each $(f,g) \in F_p$,

$$\hat{u}(f,g) = -\int_{X^t} v_{\delta,t} \mathrm{d}P_{(f,g),t};$$

$u : \Sigma \to \mathbb{R}$ is the mixed extension of $\hat{u}$: $u(\sigma_1, \sigma_2) = \int_{F_p} \hat{u} \mathrm{d}(\sigma_1 \times \sigma_2)$ for each $(\sigma_1, \sigma_2) \in \Sigma = \Sigma_1 \times \Sigma_2$. Clearly, $\hat{u}$ is continuous.

The Minmax Theorem (see Mertens (1986, p. 237))implies that

$$\max_{\sigma_1 \in \Sigma_1} \min_{\sigma_2 \in \Sigma_2} u(\sigma_1, \sigma_2) = \min_{\sigma_2 \in \Sigma_2} \max_{\sigma_1 \in \Sigma_1} u(\sigma_1, \sigma_2).$$

Kuhn's Theorem implies that

$$\max_{f \in F_1} \min_{g \in F_2} u(f,g) = \min_{g \in F_2} \max_{f \in F_1} u(f,g).$$

Hence, let $f^* \in F_1$ be such that $\min_{g\in F_2} u(f^*, g) = \max_{f\in F_1} \min_{g\in F_2} u(f, g)$.

Let $\varepsilon > 0$ be given. By (7),

$$\min_{g\in F_2}\max_{f\in F_1} u(f, g) > -\left[(M-\varepsilon)ce^{-\frac{d\varepsilon^2}{b(\delta,t)}} + \varepsilon\right]$$

and, hence,

$$-\int_{X^t} v_{\delta,t} \mathrm{d}P_{(f^*,g),t} \geq \min_{g\in F_2} u(f^*, g) > -\left[(M-\varepsilon)ce^{-\frac{d\varepsilon^2}{b(\delta,t)}} + \varepsilon\right].$$

Thus, for each $g \in F_2$,

$$\int_{X^t} v_{\delta,t} \mathrm{d}P_{(f^*,g),t} < (M-\varepsilon)ce^{-\frac{d\varepsilon^2}{b(\delta,t)}} + \varepsilon.$$

This establishes the condition (8) and concludes the proof.

A.3 Proof of Theorem 2

Theorem 3 in Blackwell (1956) shows that condition 1 in Theorem 2 is equivalent to condition 2. Furthermore, in light of Theorem 1, it suffices to show that condition 1 implies condition 5 and that condition 1 is implied by condition 3 and also by condition 4.

A.3.1 Proof that $S \cap T(q) \neq \emptyset$ for each $q \in Q$ implies that S is securable

Let $S \subseteq \mathbb{R}^N$ be convex and such that $S \cap T(q) \neq \emptyset$ for each $q \in Q$. Define let $c = |X|$ and $d = 2/B^2$ (recall that $B = |X| \max_{x\in X} ||x||$). Fix $g \in F_2$. Lemma 5 implies that there exists $f \in F_1$ such that

$$\frac{1-\delta}{1-\delta^t}\sum_{k=1}^{t}\delta^{k-1}\sum_{x\in X}\beta_x(x_1,\ldots,x_{k-1})x \in S$$

for all $t \in \mathbb{N}$, $(x_1, \ldots, x_t) \in X^t$ and $\delta \in (0, 1)$. Then, for each $\varepsilon > 0$, $\delta \in (0, 1)$ and $t \in \mathbb{N}$, Lemma 1 implies that

$$P_{(f,g),t}\left(\left\{(x_1,\ldots,x_t) \in X^t : d\left(\frac{1-\delta}{1-\delta^t}\sum_{k=1}^{t}\delta^{k-1}x_k, S\right) \geq \varepsilon\right\}\right) < ce^{-\frac{d\varepsilon^2}{b(\delta,t)}}.$$

A.3.2 Proof that $S \cap T(q) \neq \emptyset$ for each $q \in Q$ is implied by S being approachable with discounting and by S being (δ, t)-approachable for each $\delta \in (0,1)$ and $t \in \mathbb{N}$

Let S be closed and convex be such that $S \cap T(q) = \emptyset$ for some $q \in Q$. Let $g \equiv q$; it then follows by Lemma 4 that

$$\frac{1-\delta}{1-\delta^t} \sum_{k=1}^{t} \delta^{k-1} \sum_{x \in X} \beta_x(x_1, \ldots, x_{k-1}) x \in T(q)$$

for each $\delta \in (0,1)$, $t \in \mathbb{N}$, $f \in F_1$, and $(x_1, \ldots, x_t) \in X^t$. Since $S \cap T(q) = \emptyset$, then $d(z, S) > 0$ for all $z \in T(q)$ and, since $T(q)$ is compact, then $\min_{z \in T(q)} d(z, S) > 0$. Let $\eta = \min\{1, \min_{z \in T(q)} d(z, S)\}/2 > 0$.

Let, for convenience,

$$C_{\delta,t,\eta} = \left\{ (x_1, \ldots, x_t) \in X^t : \left\| \frac{1-\delta}{1-\delta^t} \sum_{k=1}^{t} \delta^{k-1} x_k - \frac{1-\delta}{1-\delta^t} \sum_{k=1}^{t} \delta^{k-1} \sum_{x \in X} \beta_x(x_1, \ldots, x_{k-1}) x \right\| < \eta \right\}.$$

Then $C_{\delta,t,\eta} \subseteq A_{\delta,t,\eta}$ since $2\eta \leq \min_{z \in T(q)} d(z, S)$ and $P_{(f,g),t}\left(C_{\delta,t,\eta}\right) \geq 1 - |X| e^{-\frac{2\eta^2}{B^2 b(\delta,t)}}$ by Lemma 1.

Fix $\varepsilon > 0$ and let $\delta^* = \delta_{\min\{\varepsilon,\eta\}}$ and $T = T_{\min\{\varepsilon,\eta\}}$ be given by Corollary 1 with $\min\{\varepsilon, \eta\}$ in place of γ. Fix $\delta \geq \delta^*$ and $t \geq T$. If $\eta \geq \varepsilon$, then

$$P_{(f,g),t}\left(A_{\delta,t,\eta}\right) \geq P_{(f,g),t}\left(C_{\delta,t,\eta}\right) \geq 1 - |X| e^{-\frac{2\eta^2}{B^2 b(\delta,t)}} \geq 1 - |X| e^{-\frac{2\varepsilon^2}{B^2 b(\delta,t)}} > 1 - \varepsilon;$$

if $\eta < \varepsilon$, then

$$P_{(f,g),t}\left(A_{\delta,t,\eta}\right) \geq P_{(f,g),t}\left(C_{\delta,t,\eta}\right) \geq 1 - |X| e^{-\frac{2\eta^2}{B^2 b(\delta,t)}} > 1 - \eta > 1 - \varepsilon.$$

Hence, $P_{(f,g),t}\left(A_{\delta,t,\eta}\right) > 1 - \varepsilon$ for each $f \in F_1$, $\delta \geq \delta^*$ and $t \geq T$.

Using the above, we next show that S is neither approachable with discounting nor (δ, t)-approachable for each $\delta \in (0,1)$ and $t \in \mathbb{N}$. Note first that, as above, we may choose $\eta < 1$. Let $\varepsilon > 0$ be such that $\eta < 1 - \varepsilon$.

If S were to be approachable with discounting, then, for some $\delta \geq \delta^*$, $f \in F_1$ and $t \geq T$, we would have that $P_{f,g,t}(A_{\delta,t,\eta}) < \eta < 1 - \varepsilon$, contradicting what has been shown above. Thus, S is not approachable with discounting.

If S were to be (δ,t)-approachable for each $\delta \in (0,1)$ and $t \in \mathbb{N}$, then, for each $\delta \in (0,1)$ and $t \in \mathbb{N}$, there exists $f \in F_1$ such that $P_{f,g,t}(A_{\delta,t,\eta}) < (M/\eta - \eta)ce^{-\frac{d\eta^4}{b(\delta,t)}} + \eta$. Let, by Lemma 2, $\delta \geq \delta^*$ and $t \geq T$ be such that $(M/\eta - \eta)ce^{-\frac{d\eta^4}{b(\delta,t)}} + \eta < 1 - \varepsilon$. Hence, $P_{f,g,t}(A_{\delta,t,\eta}) < 1 - \varepsilon$, contradicting what has been shown above. Thus, S is not (δ,t)-approachable.

A.4 Proof of Theorem 3

Let $p \in P$ be such that $R(p) \subseteq S$ and $f \equiv p$. Fix $\varepsilon > 0$ and let $\delta^* = \delta_{\varepsilon/3}$ and $T^* = T_{\varepsilon/3}$ be given by Corollary 1 with $\varepsilon/3$ in place of γ. Thus, for each $\delta \geq \delta^*$, $t \geq T^*$ and $g \in F_2$, Lemma 3 implies that

$$\frac{1-\delta}{1-\delta^t}\sum_{k=1}^{t}\delta^{k-1}\sum_{x\in X}\beta_x(x_1,\ldots,x_{k-1})x \in R(p) \subseteq S$$

for all $(x_1,\ldots,x_t) \in X^t$. Hence, Lemma 1 implies that $P_{(f,g),t}\left(A_{\delta,t,\varepsilon/3}\right) < \varepsilon/3$ and that

$$P_{(f,g),t}\left(A_{\delta,t,\varepsilon}\right) < ce^{-\frac{d\varepsilon^2}{b(\delta,t)}} < ce^{-\frac{d(\varepsilon/3)^2}{b(\delta,t)}} < \frac{\varepsilon}{3} < \varepsilon.$$

In addition, fix $\delta \geq \delta^*$ and let $T > T^*$ be given by Lemma 6 corresponding to $\varepsilon/3$ and δ. We then have that $A_{\delta,\varepsilon} \subseteq \tilde{A}_{\delta,T,\varepsilon/3}$ and $\bigcup_{t=T}^{\infty}\tilde{A}_{\delta,t,\varepsilon} \subseteq \tilde{A}_{\delta,T,\varepsilon/3}$. Thus, for each $g \in F_2$,

$$P_{(f,g),\infty}((\bigcup_{t=T}^{\infty}\tilde{A}_{\delta,t,\varepsilon}) \cup A_{\delta,\varepsilon}) \leq P_{(f,g),\infty}(\tilde{A}_{\delta,T,\varepsilon/3}) = P_{(f,g),T}(A_{\delta,T,\varepsilon/3}) < \frac{\varepsilon}{3} < \varepsilon.$$

B Folk Theorem with perfect monitoring and finite automata

To illustrate our result on approachability with discounting, we use Theorem 3 to establish a Folk Theorem with perfect monitoring and finite automata.

The stage game: A *normal form game* G is defined by $G = (A_i, u_i)_{i\in N}$, where $N = \{1,\ldots,n\}$ is a finite set of players, A_i is the set of player i's actions and $u_i : \prod_{j\in N} A_j \to \mathbb{R}$ is player i's payoff function. We assume that A_i is finite for all $i \in N$.

Let $A = \prod_{i \in N} A_i$ and $A_{-i} = \prod_{j \neq i} A_i$. We shall denote the maximum payoff in absolute value some player can obtain by $B = \max_{i \in N} \max_{a \in A} |u_i(a)|$. The set of mixed action of player $i \in N$ is denoted by Δ_i. As above, we let $\Delta = \prod_{i \in N} \Delta_i$ and $\Delta_{-i} = \prod_{j \neq i} \Delta_i$. For each $i \in N$, the mixed extension of player i's payoff function is also denoted by u_i.

For any $i \in N$ denote, respectively, the *minmax payoff* and a *minmax profile* for player i by $v_i = \min_{\sigma_{-i} \in \Delta_{-i}} \max_{a_i \in A_i} u_i(a_i, \sigma_{-i})$ and $\mu^i \in \Delta$, where $\mu^i_{-i} \in \arg\min_{\sigma_{-i} \in \Delta_{-i}} \max_{a_i \in A_i} u_i(a_i, a_{-i})$ and $\mu^i_i \in \arg\max_{a_i \in A_i} u_i(a_i, \mu^i_{-i})$.

Let $\mathcal{U} = \{u \in \text{co}(u(A)) : u_i \geq v_i \text{ for all } i \in N\}$ denote the set of *mixed individually rational payoffs* and $\mathcal{U}^0 = \{u \in \text{co}(u(A)) : u_i > v_i \text{ for all } i \in N\}$. The game G is *full-dimensional* if the interior of $\mathcal{U}$ in $\mathbb{R}^n$ is nonempty.

The repeated game: The *infinitely repeated game* consists of an infinite sequence of repetitions of G. We denote the action of any player i in the repeated game at any date $t = 1, 2, 3, \ldots$ by $a^t_i \in A_i$. Also, let $a^t = (a^t_1, \ldots, a^t_n)$ be the profile of choices at t.

For any $t \geq 1$, a *t-stage history* is a sequence $h = (a^1, \ldots, a^t) \in A^t$ (the t-fold Cartesian product of A). The set of all t -stage histories is denoted by $H_t = A^t$. We represent the initial (empty) history by H_0. The set of all histories is defined by $H = \bigcup_{t \in \mathbb{N}_0} H_t$.[7] We also denote the length of any history $h \in H$ by $\ell(h)$.

For any $a \in A$ and $k \in \mathbb{N}$, we denote a finite path consisting of a being played k times consecutively by $(a; k)$. Also, for two positive length histories $h = (a^1, \ldots, a^{\ell(h)})$ and $\bar{h} = (\bar{a}^1, \ldots, \bar{a}^{\ell(\bar{h})})$ in H we define the *concatenation of h and $\bar{h}$* by $h \cdot \bar{h} = (a^1, \ldots, a^{\ell(h)}, \bar{a}^1, \ldots, \bar{a}^{\ell(\bar{h})})$.

We assume that players may choose mixed actions but observe only the realization of those mixed actions. For all $i \in N$, a *finite automata* for player i is $f_i = (S_i, s^0_i, \tau_i, g_i)$ where S_i is a finite set of states, $s^0_i \in S$ is the initial state, $\tau_i : S \times A \to S$ is the transition function and $g_i : S \to \Delta_i$ is the behavior function; the probability of $a_i \in A_i$ being played at state $s_i \in S_i$ is denoted by $g_i(s_i)[a_i]$. The set of player i's finite automata is denoted by F_i, and $F = \prod_{i \in N} F_i$.

[7] We use $\mathbb{N}_0$ and $\mathbb{N}$ to denote, respectively, the set of non-negative and positive integers.

Given a finite automaton $f_i \in F_i$ and a history $h \in H \setminus H_0$, let s_i^h be defined by induction as follows: Letting $h = (a^1, \dots, a^{\ell(h)})$, let $s_i^k = \tau_i(s_i^{k-1}, a^k)$ for each $1 \leq k \leq \ell(h)$ and set $s_i^h = s_i^{\ell(h)}$; in the case $h = H_0$, set $s_i^h = s_i^0$. The *finite automaton induced by* f_i *at* h is $f_i|h = (S_i, s_i^h, \tau_i, g_i)$. We will use $f|h$ to denote $(f_1|h, \dots, f_n|h)$ for every $f = (f_1, \dots, f_n) \in F$ and $h \in H$.

Any finite automata $f \in F$ induces, for every period $t \in \mathbb{N}$, a probability distribution $\tilde{\pi}^t(f)$ over pure actions and a probability distribution $P_{f,t}$ over H_t as follows: $\tilde{\pi}^1(f)[a] = P_{f,1}(a) = g(s^0)[a] = \prod_{i \in N} g_i(s_i^0)[a_i]$ for all $a \in A = H_1$ and, for any $t > 1$, $h \in H_t$ and $a \in A$, letting $h = \bar{h} \cdot \bar{a}$ with $\bar{h} \in H_{t-1}$, $P_{f,t}(h) = P_{f,t-1}(\bar{h}) g(s^{\bar{h}})[\bar{a}]$ and $\tilde{\pi}^t(f)[a] = \sum_{h \in H_{t-1}} P_{f,t}(h \cdot a)$.

We assume that all players discount the future payoffs by a common discount factor $\delta \in (0,1)$. Thus, the *payoff in the repeated game* is given by $U_i(f, \delta) = (1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} \sum_{a \in A} u_i(a) \tilde{\pi}^t(f)[a]$ for all $i \in N$ (when the meaning is clear we will refer to repeated game payoff by $U_i(f)$ without an explicit reference to δ).

We denote the repeated game described above for discount factor $\delta \in (0,1)$ by $G^\infty(\delta)$. A finite automata $f \in F$ is a *Nash equilibrium* of $G^\infty(\delta)$ if $U_i(f) \geq U_i(\hat{f}_i, f_{-i})$ for all $i \in N$ and $\hat{f}_i \in F_i$. Also, $f \in F$ is a *subgame-perfect equilibrium* (SPE henceforth) of $G^\infty(\delta)$ if $f|h$ is a Nash equilibrium for all $h \in H$.

The following is our Folk Theorem for finite automata.

Theorem 4 *Let G be a full-dimensional n-player game. Then, for all $\varepsilon > 0$, there exist $\delta^* \in (0,1)$ such that, for all $u \in \mathcal{U}$ and $\delta \geq \delta^*$, there exists a finite automata SPE $f \in F$ of $G^\infty(\delta)$ such that $\|U(f,\delta) - u\| < \varepsilon$.*

B.1 Proof of Theorem 4

For all $x \in \mathbb{R}^n$, let $||x|| = \max_{i=1,\dots,n} |x_i|$. Since $\mathcal{U}$ is compact, it suffices to show that for all $\varepsilon > 0$ and all $u \in \mathcal{U}$, there exists $\delta^* \in (0,1)$ such that for all $\delta \geq \delta^*$, there exists a finite automata SPE f of $G^\infty(\delta)$ with $\|U(f,\delta) - u\| < \varepsilon$. Furthermore, since $\mathcal{U}$ equals the closure of $\mathcal{U}^0$, we only need to show that the above holds for any $u \in \mathcal{U}^0$. Therefore, we show that for all $\varepsilon > 0$ and $u \in \mathcal{U}^0$, there exists $\delta^* \in (0,1)$ such that for all $\delta \geq \delta^*$, there exists a finite automata SPE f of $G^\infty(\delta)$ with $\|U(f,\delta) - u\| < \varepsilon$.

For convenience, we normalize payoffs so that $v_i = 0$ for all $i \in N$.

B.2 Preliminary results

We make the following construction, analogous to Gossner (1995), for each $\delta \in (0,1)$, $t \in \mathbb{N}$ and $\eta > 0$, and then show below that these parameters can be chosen to have certain desirable properties.

For each $i, d \in N$ with $i \neq d$, $a \in A$ and $\hat{h} = (\hat{a}^1, \ldots, \hat{a}^t) \in H_t$, let

$$n(a, \hat{h}) = \sum_{k=1}^{t} \delta^{k-1} 1_a(\hat{a}^k),$$
$$n(a_{-i}, \hat{h}) = \sum_{b_i \in A_i} n((b_i, a_{-i}), \hat{h}),$$
$$\Phi_i(d, \hat{h}) = \frac{1-\delta}{1-\delta^t} \sum_{a \in A} |n(a, \hat{h}) - n(a_{-i}, \hat{h}) \mu_i^d(a_i)|,$$

and

$$\alpha_i(d, \hat{h}, \eta) = \begin{cases} 1 & \text{if } \Phi_i(d, \hat{h}) < \eta, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 7 *For every $0 < \varepsilon_1 < 1$, there exists $\eta > 0$ such that, for every $d \in N$, $\delta \in (0,1)$, $t \in \mathbb{N}$ and $\hat{h} = (\hat{a}^1, \ldots, \hat{a}^t) \in H_t$ such that $\alpha_i(d, \hat{h}, \eta) = 1$ for all $i \neq d$,*

$$\frac{1-\delta}{1-\delta^t} \sum_{k=1}^{t} \delta^{k-1} u_d(\hat{a}^k) < \varepsilon_1.$$

Proof. Let $0 < \varepsilon_1 < 1$. Moreover, let $\eta > 0$ be such that $(n-1)B|A|^2\eta < \varepsilon_1$.

Consider $d \in N$, $\delta \in (0,1)$, $t \in \mathbb{N}$ and $\hat{h} = (\hat{a}^1, \ldots, \hat{a}^t) \in H_t$ such that $\alpha_i(d, \hat{h}, \eta) = 1$ for all $i \neq d$. First, we reorder the players such that the player to be punished is called player n, i.e. $d = n$. Second, we write, for each $a \in A$, $n(a)$ instead of $n(a, \hat{h})$. Then, for every $a \in A$ and every $i \neq n$:

$$|n(a) - n(a_{-i})\mu_i^n(a_i)| < \eta \frac{1-\delta^t}{1-\delta}.$$

Fix $a = (a_1, a_2, \ldots, a_n) \in A$. In particular, we have that

$$|n(a) - \sum_{b_1 \in A_1} n(b_1, a_2, \ldots, a_n)\mu_1^n(a_1)| < \eta \frac{1-\delta^t}{1-\delta}.$$

Since, for every $b_1 \in A_1$,

$$|n(b_1, a_{-1}) - \sum_{b_2 \in A_2} n(b_1, b_2, a_3, \ldots, a_n)\mu_2^n(a_2)| < \eta\frac{1-\delta^t}{1-\delta},$$

we obtain:

$$|n(a) - \sum_{(b_1,b_2) \in A_1 \times A_2} n(b_1, b_2, a_3, \ldots, a_n)\mu_1^n(a_1)\mu_2^n(a_2)| < 2|A|\eta\frac{1-\delta^t}{1-\delta}.$$

Repeating the same procedure $n-1$ times implies that

$$\left| n(a) - \sum_{(b_1,\ldots,b_{n-1}) \in A_{-n}} n(b_1, b_2, \ldots, b_{n-1}, a_n) \prod_{j=1}^{n-1} \mu_j^n(a_j) \right| < (n-1)|A|\eta\frac{1-\delta^t}{1-\delta}.$$

Hence,

$$\frac{1-\delta}{1-\delta^t}|n(a) - \sum_{(b_1,\ldots,b_{n-1}) \in A_{-n}} n(b_1, b_2, \ldots, b_{n-1}, a_n) \prod_{j=1}^{n-1} \mu_j^n(a_j)| < (n-1)|A|\eta. \tag{9}$$

As $a \in A$ is arbitrary, it follows that (9) holds for all $a \in A$.

Define, for each $a_n \in A_n$,

$$r_n(a_n) = \frac{1-\delta}{1-\delta^t} \sum_{b_{-n} \in A_{-n}} n(b_{-n}, a_n).$$

We then have that $r_n \in \Delta_n$ since $\sum_{a_n} r_n(a_n) = 1$. It follows from the definition of r_n and (9) that, for all $a \in A$,

$$\left| \frac{1-\delta}{1-\delta^t} n(a) - r_n(a_n) \prod_{j=1}^{n-1} \mu_j^n(a_j) \right| < (n-1)|A|\eta.$$

Hence,

$$\frac{1-\delta}{1-\delta^t} \sum_{a \in A} n(a)u_n(a) < u_n(r_n, \mu_{-n}^n) + B(n-1)|A|^2\eta < \varepsilon_1,$$

and, therefore,

$$\frac{1-\delta}{1-\delta^t} \sum_{k=1}^{t} \delta^{k-1} u_n(\hat{a}^k) = \frac{1-\delta}{1-\delta^t} \sum_{a \in A} n(a)u_n(a) < \varepsilon_1.$$

This concludes the proof. ■

Fix $i, d \in N$ with $i \neq d$ and let $\tilde{\mu}_i^d$ be player i's strategy consisting of playing μ_i^d each period independently of the history. Given a strategy σ_{-i} for the remaining players and $t \in \mathbb{N}$, let $P_{(\tilde{\mu}_i^d, \sigma_{-i}),t}$ be the probability measure on H_t induced by $(\tilde{\mu}_i^d, \sigma_{-i})$.

Given $c \in (0,1)$, for each $\delta \in (0,1)$, let $t(c,\delta) = 1$ if $\delta \leq c$ and, if $\delta > c$, let $t(c,\delta)$ be the highest integer $t \in \mathbb{N}$ such that $\delta^t \geq c$. Hence, $|\delta^{t(c,\delta)} - c| < (1-\delta)/\delta$ whenever $\delta > c$ and, therefore, $\lim_{\delta \to 1} \delta^{t(c,\delta)} = c$.

Lemma 8 *For all $c \in (0,1)$, $\eta > 0$ and $\varepsilon_2 > 0$, there exists $\bar{\delta} \in (0,1)$ such that, for all $\delta \geq \bar{\delta}$,*

$$P_{(\tilde{\mu}_i^d, \sigma_{-i}),t(c,\delta)}\left(\{\hat{h} \in H_{t(c,\delta)} : \Phi_i(d,\hat{h}) \geq \eta\}\right) < \varepsilon_2$$

for all $i, d \in N$ with $i \neq d$ and $\sigma_{-i} \in F_{-i}$.

Proof. This result will be a consequence of Theorem 3. Let $c \in (0,1)$, $\eta > 0$ and $\varepsilon_2 > 0$ be given. Note first that it is enough to show that, for each $i, d \in N$ with $i \neq d$, there exists $\bar{\delta}_{i,d} \in (0,1)$ such that, for all $\delta \geq \bar{\delta}_{i,d}$,

$$P_{(\tilde{\mu}_i^d, \sigma_{-i}),t(c,\delta)}\left(\{\hat{h} \in H_{t(c,\delta)} : \Phi_i(d,\hat{h}) \geq \eta\}\right) < \varepsilon_2$$

for all $\sigma_{-i} \in F_{-i}$. Indeed, the conclusion of the lemma will follow by letting $\bar{\delta} = \max_{i,d:i\neq d} \bar{\delta}_{i,d}$.

Fix $i, d \in N$ such that $i \neq d$. We embed A in $\mathbb{R}^{|A|}$ by, first, letting $\theta : A \to \{1, \ldots, |A|\}$ be 1-1 and onto and, second, letting $x(a) \in \mathbb{R}^{|A|}$ be such that

$$x_l(a) = \begin{cases} 1 & \text{if } l = \theta(a), \\ 0 & \text{otherwise.} \end{cases}$$

Let $X = \{x(a) : a \in A\}$ and $m(a) = 1_{x(a)}$ for all $a \in A$. Consider a 2-player game where player I's action set is A_i and player II's action set is A_{-i}, and hence $P = \Delta_i$ and $Q = \Delta_{-i}$. Let $p \in P$ be defined by $p_{a_i} = \mu_i^d(a_i)$ for all $a_i \in A_i$ and let $\Lambda_i : \mathrm{co}(X) \to \mathbb{R}$ be defined by

$$\Lambda_i(z) = \sum_{a \in A} \left| z_a - \sum_{b_i \in A_i} p_{a_i} z_{(b_i, a_{-i})} \right|$$

for each $z \in \mathrm{co}(X)$. Since Λ_i is continuous and $\mathrm{co}(X)$ is compact, there exists $0 < \varepsilon < \varepsilon_2$ such that

$$||z - z'|| < \varepsilon \text{ and } z, z' \in \mathrm{co}(X) \text{ imply } |\Lambda_i(z) - \Lambda_i(z')| < \eta. \tag{10}$$

We have that $R(p) = \{z \in \mathrm{co}(X) : \Lambda_i(z) = 0\}$. Indeed, if $z \in R(p)$, then $z = \sum_{a_{-i}} q_{a_{-i}} \sum_{a_i} p_{a_i} x(a)$ for some $q \in Q$. Thus, for each $a \in A$, $z_a = p_{a_i} q_{a_{-i}}$ and, therefore, $\Lambda_i(z) = \sum_{a\in A} \left|p_{a_i} q_{a_{-i}} - \sum_{b_i \in A_i} p_{a_i} p_{b_i} q_{a_{-i}}\right| = \sum_{a\in A} \left|p_{a_i} q_{a_{-i}} - p_{a_i} q_{a_{-i}}\right| = 0$. Conversely, let $z \in \mathrm{co}(X)$ be such that $\Lambda_i(z) = 0$. Since $z \in \mathrm{co}(X)$, then $z = \sum_a \lambda_a x(a)$ for some $\{\lambda_a\}_{a\in A} \subset \mathbb{R}_+$ with $\sum_a \lambda_a = 1$ and, since $\Lambda_i(z) = 0$, then $z_a = \sum_{b_i\in A_i} p_{a_i} z_{(b_i, a_{-i})}$ for each $a \in A$. Define $q_{a_{-i}} = \sum_{b_i \in A_i} \lambda_{(b_i, a_{-i})}$ for all $a_{-i} \in A_{-i}$ and note that $q \in Q$. Since $z_a = \lambda_a$ for all $a \in A$ given the definition of $x(a)$, it follows from $z_a = p_{a_i} \sum_{b_i \in A_i} z_{(b_i, a_{-i})}$ for each $a \in A$ that $\lambda_a = p_{a_i} q_{a_{-i}}$ for each $a \in A$. Thus, $z \in R(p)$.

It then follows by Theorem 3 that, with $f \equiv p$, there exists $\delta^* \in (0, 1)$ and $T \in \mathbb{N}$ such that

$$P_{(f,g),t}\left(\left\{(x_1, \ldots, x_t) \in X^t : d\left(\frac{1-\delta}{1-\delta^t} \sum_{k=1}^{t} \delta^{k-1} x_k, R(p)\right) \geq \varepsilon\right\}\right) < \varepsilon \tag{11}$$

holds for each $\delta \geq \delta^*$, $t \geq T$ and $g \in F_{II}$.

Let $\gamma > 0$ be such that $1 - c > \gamma$ and let $\bar{\delta}_{i,d} \in (\delta^*, 1)$ be such that $\delta > c$ and $1 - \delta^T < \gamma$ for each $\delta \geq \bar{\delta}_{i,d}$. Thus, $\delta^T > 1 - \gamma > c$ and, hence, $t(c, \delta) \geq T$ for each $\delta \geq \bar{\delta}_{i,d}$. Thus, for each $\delta \geq \bar{\delta}_{i,d}$ and $g \in F_{II}$, we have that $\delta > \delta^*$ and $t(c, \delta) \geq T$, and (11) implies that

$$P_{(f,g),t(c,\delta)}\left(\left\{d\left(\frac{1-\delta}{1-\delta^{t(c,\delta)}} \sum_{k=1}^{t(c,\delta)} \delta^{k-1} x_k, R(p)\right) \geq \varepsilon\right\}\right) < \varepsilon.$$

Thus, by (10),

$$P_{(f,g),t(c,\delta)}\left(\left\{\Lambda_i\left(\frac{1-\delta}{1-\delta^{t(c,\delta)}} \sum_{k=1}^{t(c,\delta)} \delta^{k-1} x_k\right) \geq \eta\right\}\right) < \varepsilon < \varepsilon_2.$$

Let $\sigma_{-i} \in F_{-i}$ be given. Define $g \in F_{II}$ as follows: for all $t \in \mathbb{N}$ and $(x_1, \ldots, x_t) \in X^t$, let, for each $1 \leq k \leq t$, $a_k \in A$ be such that $x_k = x(a_k)$ and set $g(x_1, \ldots, x_t) =$

$\sigma_{-i}(a_1, \ldots, a_t)$. Hence, we have that

$$P_{(\tilde{\mu}_i^d, \sigma_{-i}), t(c,\delta)}(\hat{a}_1, \ldots, \hat{a}_{t(c,\delta)}) = P_{(f,g), t(c,\delta)}(x(\hat{a}_1), \ldots, x(\hat{a}_{t(c,\delta)})).$$

Furthermore, note that, for each $\hat{h} = (\hat{a}_1, \ldots, \hat{a}_{t(c,\delta)}) \in H_{t(c,\delta)}$,

$$\Phi_i(d, \hat{h}) = \Lambda_i \left(\frac{1-\delta}{1-\delta^{t(c,\delta)}} \sum_{k=1}^{t(c,\delta)} \delta^{k-1} x(\hat{a}_k) \right).$$

Hence, it follows that

$$P_{(\tilde{\mu}_i^d, \sigma_{-i}), t(c,\delta)} \left(\{ \hat{h} \in H_{t(c,\delta)} : \Phi_i(d, \hat{h}) \geq \eta \} \right) < \varepsilon_2,$$

as desired. ■

B.3 Parametrization

Fix any $\varepsilon > 0$ and any $u \in \mathcal{U}^0$. Since G is full-dimensional, $\mathcal{U}^0$ equals the closure of $\text{int}(\mathcal{U}^0)$ and, therefore, we may assume that $u \in \text{int}(\mathcal{U}^0)$. Let $u' \in \text{int}(\mathcal{U}^0)$ such that $u' < u$, and $\rho > 0$ be such that (i) $u'_i + \rho < u_i$ for all $i \in N$ and (ii) $||\hat{u} - u'|| \leq \rho$ implies $\hat{u} \in \mathcal{U}^0$.

Let $\varepsilon_1 > 0$ such that

$$\varepsilon_1 < \min_{d \in N} u'_d. \tag{12}$$

Let $\eta > 0$ be as in Lemma 7, corresponding to ε_1 just defined. Let $0 < \varepsilon_2 < 1$ be such that

$$(1 - 2\varepsilon_2)^{n-1} \varepsilon_1 + \left(1 - (1 - 2\varepsilon_2)^{n-1}\right) B < \min_{d \in N} u'_d. \tag{13}$$

Define

$$\bar{\varepsilon} = (1 - 2\varepsilon_2)^{n-1} \varepsilon_1 + \left(1 - (1 - 2\varepsilon_2)^{n-1}\right) B.$$

Let $c \in (0, 1)$ be such that

$$c\rho\varepsilon_2 > (1 - c)2B. \tag{14}$$

Let $\bar{\delta}$ be as in Lemma 8, corresponding to ε_2, η and c just defined.

Define $\xi > 0$ to be such that

$$2\xi < \varepsilon, \tag{15}$$

$$(1-c)(\min_d u'_d - \bar{\varepsilon}) > (1+c)2\xi, \tag{16}$$

$$c(\rho\varepsilon_2 - 4\xi) > (1-c)2B, \tag{17}$$

Such $\xi > 0$ exists due to $\varepsilon > 0$, (13) and (14), respectively.

For all $i = 1, \ldots, n$ and $\beta \in \mathbb{R}^n$, let $u^i(\beta)$ be defined by $u^i_i(\beta) = u'_i$ and $u^i_j(\beta) = u'_j + \beta_j \rho$. Furthermore, define

$$W_i = \{u^i(\beta) : \beta_j \in \{0, 1\} \text{ for all } j \in N \setminus \{i\}\}.$$

By our choice of ρ (specifically, by (ii) above), then $W_i \subseteq \mathcal{U}^0$. Define $\hat{W} = \cup_{i=1}^n W_i$. Since $\hat{W}$ is finite, order $\hat{W} = \{u^1, \ldots, u^{\bar{\omega}}\}$, where $\bar{\omega} = |\hat{W}|$. For notational convenience, let $u^0 = u$ and $W = \hat{W} \cup \{u^0\}$.

For all $k \in \mathbb{N}$, let $\mathcal{V}_k$ be the set of $u' \in \text{co}(u(A))$ such that $u' = \sum_{a \in A} p_a u(a)/k$ for some $\{p_a\}_{a \in A}$ satisfying $p_a \in \mathbb{N}_0$ and $\sum_{a \in A} p_a = k$. Using an analogous argument to Sorin (1992, Proposition 1.3), it follows that $\mathcal{V}_k$ converges to $\text{co}(u(A))$. Therefore, let $K \in \mathbb{N}$ such that

$$\text{co}(u(A)) \subseteq \cup_{x \in \mathcal{V}_K} B_\xi(x). \tag{18}$$

For all $\omega \in \{0, \ldots, \bar{\omega}\}$, let $x^\omega \in \mathcal{V}_K$ be such that

$$||x^\omega - u^\omega|| < \xi \tag{19}$$

and $\{p^\omega_a\}_{a \in A}$ be such that $\frac{1}{K} \sum_{a \in A}^K p^\omega_a u(a) = x^\omega$. Letting $A = \{a^1, \ldots, a^r\}$ where $r = |A|$, define, for each $\omega \in \{0, \ldots, \bar{\omega}\}$, $\hat{\pi}^{(\omega)} = (\hat{\pi}^{(\omega),1}, \hat{\pi}^{(\omega),2}, \ldots)$ as the repetition of the cycle

$$(\underbrace{a^1, \ldots, a^1}_{p^\omega_{a^1} \text{ times}}, \ldots, \underbrace{a^r, \ldots, a^r}_{p^\omega_{a^r} \text{ times}}).$$

In the construction below, $\hat{\pi}^{(\omega)}$ will be the equilibrium path when $\omega = 0$ and a "reward path" when $\omega > 0$.

Let $\delta^* \in [\bar{\delta}, 1)$ be such that for all $\delta \geq \delta^*$, letting $t(c, \delta)$ be as in Lemma 8,

$$\sup_{x\in[-B,B]^K}\left|\frac{1-\delta}{1-\delta^K}\sum_{k=1}^{K}\delta^{k-1}x^k-\frac{1}{K}\sum_{k=1}^{K}x^k\right|<\xi, \tag{20}$$

$$\delta^{t(c,\delta)}(\rho\varepsilon_2-4\xi)>(1-\delta^{t(c,\delta)})2B, \text{ and} \tag{21}$$

$$u'_d-2\xi>(1-\delta)B+\delta(1-\delta^{t(c,\delta)})\bar{\varepsilon}+\delta^{1+t(c,\delta)}(u'_d+2\xi) \text{ for all } d\in N. \tag{22}$$

Note that such $\delta^*\in(0,1)$ exists because of (16) and (17), and because the limit of the left hand side of (20) as $\delta\to 1$ is 0.

Fix any $\delta\geq\delta^*$ and set $T=t(c,\delta)$. We will now demonstrate the result by constructing a finite automata SPE f with $||U(f)-u||<\varepsilon$.

B.4 Punishment play

Next we define the mixed actions to be played during the punishment phases.

Let $C:W\to\mathbb{R}^n$ be defined by setting, for all $0\leq\omega\leq\bar{\omega}$,

$$C(u^\omega)=\frac{1-\delta}{1-\delta^K}\sum_{k=1}^{K}\delta^{k-1}u(\hat{\pi}^{(\omega),k}).$$

We next define a function $w:N\times H_T\to C(W)$ that determines the reward payoff after a punishment phase. Let $d\in N$ and $\hat{h}=(\hat{a}^1,\ldots,\hat{a}^T)\in H_T$. Set, for all $j\neq d$,

$$\beta_j(d,\hat{h})=\alpha_j(d,\hat{h},\eta).$$

Also, set

$$w(d,\hat{h})=C(u^d(\beta(d,\hat{h}))).$$

Let $\sigma^*:N\times\cup_{t=0}^{T-1}H_t\to\Delta$ and $V^*:N\times\cup_{t=0}^{T-1}H_t\to\mathbb{R}^n$ be such that the following property holds: For all $d\in N$, $0\leq t\leq T-1$, $\hat{h}\in H_t$ and $i\in N$, then:

(a) If $\ell(\hat{h})=T-1$, then $\sigma_i^*(d,\hat{h})$ solves

$$\max_{\sigma_i\in\Delta_i}[(1-\delta)u_i(\sigma_i,\sigma_{-i}^*(d,\hat{h}))+\delta\sum_{a\in A}(\sigma_i,\sigma_{-i}^*(d,\hat{h}))[a]w_i(d,\hat{h}\cdot a)]$$

and

$$V_i^*(d,\hat{h})=(1-\delta)u_i(\sigma^*(d,\hat{h}))+\delta\sum_{a\in A}\sigma^*(d,\hat{h})[a]w_i(d,\hat{h}\cdot a).$$

(b) If $\ell(\hat{h}) < T-1$, then $\sigma_i^*(d,\hat{h})$ solves

$$\max_{\sigma_i \in \Delta_i} [(1-\delta) u_i(\sigma_i, \sigma_{-i}^*(d,\hat{h})) + \delta \sum_{a \in A} (\sigma_i, \sigma_{-i}^*(d,\hat{h}))[a] V_i^*(d, \hat{h}\cdot a)]$$

and

$$V_i^*(d,\hat{h}) = (1-\delta) u_i(\sigma^*(d,\hat{h})) + \delta \sum_{a \in A} \sigma^*(d,\hat{h})[a] V_i^*(d,\hat{h}\cdot a).$$

The existence of σ^* and V^* can be established using, for each fixed $d \in N$, backwards induction and Nash's existence theorem.

For notational convenience, for all $d \in N$, let $\sigma^d = \sigma^*(d,\cdot)$.

Lemma 9 *For all $i, d \in N$, with $i \neq d$,*

$$P_{\sigma^d,T}\left(\{\hat{h} \in H_T : \Phi_i(d,\hat{h}) \geq \eta\}\right) < 2\varepsilon_2.$$

Proof. Suppose not. Then, for some $i, d \in N$ with $i \neq d$,

$$P_{\sigma^d,T}\left(\{\hat{h} \in H_T : \Phi_i(d,\hat{h}) \geq \eta\}\right) \geq 2\varepsilon_2.$$

Consider strategy $\tilde{\mu}_i^d$ for player i and let $V_i(d,H_0)$ be player i's expected discounted payoff from $(\tilde{\mu}_i^d, \sigma_{-i}^d)$, i.e.

$$V_i(d,H_0) = (1-\delta) \sum_{\hat{h}=(\hat{a}^1,\ldots,\hat{a}^T)\in H_T} P_{(\tilde{\mu}_i^d, \sigma_{-i}^d),T}(\hat{h}) \left(\sum_{k=1}^{T} \delta^{t-1} u_i(\hat{a}^k) + \delta^T w_i(d,\hat{h}) \right).$$

Given the definition of σ^*, we have that

$$V_i^*(d,H_0) \geq V_i(d,H_0). \tag{23}$$

Furthermore, by Lemma 8, (19) and (20),

$$V_i(d,H_0) \geq -B(1-\delta^T) + \delta^T(u_i' - 2\xi) + \delta^T \rho(1-\varepsilon_2).$$ [8]

By (19) and (20),

$$V_i^*(d,H_0) \leq B(1-\delta^T) + \delta^T(u_i' + 2\xi) + \delta^T \rho(1-2\varepsilon_2).$$

[8] It is interesting to note that, since σ_{-i}^d is fixed, it would suffice in Lemma 8 to establish that, for each σ_{-i}, player i has a strategy σ_i such that he, player i, passes the test with a probability of at least $1-\varepsilon_2$. This could be achieved by using Theorem 2 to show that $R(p)$ is securable.

Hence, by (21),

$$V_i^*(d, H_0) - V_i(d, H_0) \leq 2B(1 - \delta^T) + 4\xi\delta^T - \rho\varepsilon_2\delta^T < 0.$$

But this contradicts (23). ■

B.5 The finite automata

We next define the finite automata f, which is such that all players have a common set of states S, a common initial state s^0 and a common transition function $\tau : S \times A \to S$. The set of states is $S = (W \times \{1, \dots, K\}) \cup \left(N \times \cup_{t=0}^{T-1} H_t\right)$, with initial state $s^0 = (u^0, 1)$. The transition function τ is defined as follows: First, let $D_i(a) = \{(a'_i, a_{-i}) \in A : a'_i \neq a_i\}$ for each $i \in N$ and $a \in A$. Then, for each $(s, a) \in S \times A$,

$$\tau(s,a) = \begin{cases} (u^\omega, (k+1) \text{mod} K) & \text{if } s = (u^\omega, k) \in W \times \{1, \dots, K\} \text{ and } a = \hat{\pi}^{(\omega),k}, \\ (d, H_0) & \text{if } s \in W \times \{1, \dots, K\}, \text{ and } a \in D_d(\hat{\pi}^{(\omega),k}) \text{ for some } d \in N, \\ (d, h \cdot a) & \text{if } s = (d, h) \in N \times \cup_{t=0}^{T-2} H_t, \\ (u^d(\beta(d, h \cdot a)), 1) & \text{if } s = (d, h) \in N \times H_{T-1}. \end{cases}$$

Finally, the behavior function is $g = (g_1, \dots, g_n) : S \to \Delta$ defined by

$$g(s) = \begin{cases} \hat{\pi}^{(\omega),k} \text{ if } s = (u^\omega, k) \in W \times \{1, \dots, K\}, \\ \sigma^d(h) \text{ if } s = (d, h) \in N \times \cup_{t=0}^{T-1} H_t. \end{cases}$$

Let $f = (S, s^0, \tau, g)$.

We have that $U(f, \delta) = \frac{1-\delta}{1-\delta^K} \sum_{k=1}^{K} \delta^{k-1} u(\hat{\pi}^{(0),k})$ and, by (19), (20) and (15), $||U(f, \delta) - u|| < 2\xi < \varepsilon$.

Let $f|s = (S, s, \tau, g)$ for each $s \in S$; $f|s$ is the automata obtained by changing the initial state from s^0 to s. To complete the proof of the theorem, we next establish the following for all $s \in S$, $d \in N$ and $a_d \in A_d$:

$$U_d(f|s) \geq (1 - \delta) u_d(a_d, g_{-d}(s)) + \delta U_d(f|\tau(s, (a_d, g_{-d}(s)))). \tag{24}$$

By construction, (24) holds for each $s \in N \times \cup_{t=0}^{T-1} H_t$. Consider then the case $s \in W \times \{1, \dots, K\}$. In this case, the left-hand side of (24) is, by (19) and (20), greater or equal to $u'_d - 2\xi$. By Lemmas 7 and 9, the right-hand side of (24) is less than or equal to

$$(1-\delta)B + \delta(1-\delta^T)\bar{\varepsilon} + \delta^{T+1}(u'_d + 2\xi).$$

Thus, by (22), (24) holds.

C Discounted repeated two-person games with incomplete information

In this section we consider discounted repeated two-person games with incomplete information. We use our approachability result to obtain a "punishment" strategy that holds the payoff of the other player below a certain payoff profile (one for each of his possible types). Our main result of this section shows the existence of such strategies when the upper bound on payoffs is individually rational as in in Hörner and Lovo (2009).

The setting is as described in Hörner and Lovo (2009). There are two players, 1 and 2. The set of player i's action is A_i, $i = 1, 2$, and is finite. There is a $J \times K$ array of payoff functions $u^{jk} : A \to \mathbb{R}^2$; player 1 is told the value of $j \in \{1, \ldots, J\}$ in period 1 but not that of $k \in \{1, \ldots, K\}$; player 2 is told the value of k in period 1 but not that of j. Let $T_1 = \{1, \ldots, J\}$ and $T_2 = \{1, \ldots, K\}$ be the set of types of each player.

Players select a mixed action profile in each period $t = 1, 2, \ldots$. Realized actions are observable, mixed action and realized rewards are not. Histories are, therefore, sequences of actions: $H_t = A^t$. A strategy for player i, $i = 1, 2$, is $f_i : T_i \times H \to \Delta(A_i)$. Let F_i be the set of strategies for each type of player i, i.e. the set of all functions mapping H into $\Delta(A_i)$. The set of strategies profiles is then $F_1^J \times F_2^K$. Given a common discount factor $\delta \in (0, 1)$, a type profile (j, k) and a strategy profile $f \in F_1^J \times F_2^K$, $(f_1^j, f_2^k) \in F_1 \times F_2$ induces a probability measure $P_{(f_1^j, f_2^k),\infty}$ on A^∞ as in Section 2; player i's payoff is then

$$v_i^{jk}(f, \delta) = (1-\delta) \int_{A^\infty} \sum_{k=1}^{\infty} \delta^{k-1} u_i^{jk}(a^k) \mathrm{d}P_{(f_1^j, f_2^k),\infty}(a^\infty).$$

Since $v_i^{jk}(f,\delta)$ only depends on f_1^j and f_2^k, we also let $v_i^{jk}(f_1^j, f_2^k, \delta) = v_i^{jk}(f_1, f_2, \delta)$. If player 1 of type j has belief $p_j = (p_{j,1}, \ldots, p_{j,K})$ about player 2's types, then the payoff of type j of player 1 is

$$\sum_{k=1}^{K} p_{j,k} v_i^{jk}(f,\delta),$$

and analogously for type k of player 2.

Approachability is used in repeated games with incomplete information to show the existence of strategies for each type of player 2 that guarantee that the player 1's payoff is below a certain target for each of player 1's types; such strategy of player 2 is then used to punish player 1 in case the latter deviates from some agreed upon path. Following Hörner and Lovo (2009), the payoff vectors $v_1^k = (v_1^{1k}, \ldots, v_1^{Jk}) \in \mathbb{R}^J$ that are feasible as targets for type k of player 2 for all discount factors sufficiently close to 1 are those that are *strictly individually rational* in the sense that, for each $p \in \Delta(\{1, \ldots, J\})$,

$$p \cdot v_1^k > b_1^k(p),$$

where $b_1^k(p)$ is the value of the two-person zero-sum game with payoff matrix $p \cdot u_1^k$. Indeed, using our Theorem 2, we obtain the following result.

Theorem 5 *Let $k \in \{1, \ldots, K\}$. If v_1^k is strictly individually rational, then there exists $\delta^* \in (0,1)$ such that, for each $\delta \geq \delta^*$, there exists $f_2^k \in F_2$ such that $v_1^{jk}(f_1, f_2^k, \delta) < v_1^{jk}$ for all $j \in \{1, \ldots, J\}$ and $f_1 \in F_1$.*

This result is stated without proof as part of the construction in Hörner and Lovo (2009), in the specification of the actions during the punishment phases on p. 464 and in the specification of equilibrium strategies on p. 477.

To see how Theorem 2 applies, let $X = u_1^k(A) \subseteq \mathbb{R}^J$ and $m(a) = 1_{u_1^k(a)}$ (i.e. the Dirac measure at $u_1^k(a) \in \mathbb{R}^J$).

Lemma 10 *Let $k \in \{1, \ldots, K\}$. If $v_1^k \in \mathbb{R}^J$ is strictly individually rational, then there exists $\eta > 0$ such that $\{y \in \mathbb{R}^J : y_j \leq v_1^{jk} - \eta$ for each $j = 1, \ldots, J\}$ is approachable with discounting by player 2.*

Proof. Let $\eta > 0$ be such that $p \cdot v_1^k - \eta > b_1^k(p)$ for each $p \in \Delta(\{1, \ldots, J\})$ (note that $p \mapsto p \cdot v_1^k - b_1^k(p)$ is continuous). Denote $S = \{y \in \mathbb{R}^J : y_j \leq v_1^{jk} - \eta \text{ for each } j = 1, \ldots, J\}$ and suppose that S is not approachable with discounting by player 2. It follows by Theorem 2 that there exists $p_0 \in \Delta(A_1)$ such that $R(p_0) \cap S = \emptyset$.

By the Separating Hyperplane Theorem (S is closed, $R(p_0)$ is compact, and both are nonempty and convex), there exists $\pi \in \mathbb{R}^J \setminus \{0\}$ and $c \in \mathbb{R}$ such that

$$\min_{q \in \Delta(A_2)} \pi \cdot u_1^k(p_0, q) > c > \pi \cdot y \text{ for each } y \in S,$$

noting that $R(p_0) = \{u_1^k(p_0, q) : q \in \Delta(A_2)\}$. We may assume that $||\pi||_1 = \sum_{j=1}^J |\pi_j| = 1$ (if not, replace π with $\pi/||\pi||_1$). Moreover, $\pi_j \geq 0$ for each $j = 1, \ldots, J$; if instead $\pi_j < 0$ for some $j = 1, \ldots, J$, consider, for each $j' \neq j$, $y_{j'} = v_1^{j'k} - \eta$ and let $y_j \to -\infty$ to obtain $y \in S$ and $\pi \cdot y \to \infty$.

Thus, $\pi \in \Delta(\{1, \ldots, J\})$ and

$$\min_{q \in \Delta(A_2)} \pi \cdot u_1^k(p_0, q) > c > \pi \cdot v_1^k - \eta > b_1^k(\pi) = \max_{p \in \Delta(A_1)} \min_{q \in \Delta(A_2)} \pi \cdot u_1^k(p, q) \geq \min_{q \in \Delta(A_2)} \pi \cdot u_1^k(p_0, q),$$

a contradiction. This contradiction shows that S is approachable with discounting by player 2. ■

The argument to establish Theorem 5 is completed using a result analogous to Cripps and Thomas (2003, Result 2, p. 440), which we establish using Theorem 2.

Lemma 11 *Let $k \in \{1, \ldots, K\}$. If $v_1^k \in \mathbb{R}^J$ and $\eta > 0$ are such that $\{y \in \mathbb{R}^J : y_j \leq v_1^{jk} - \eta \text{ for each } j = 1, \ldots, J\}$ is approachable with discounting by player 2, then there exists $\delta^* \in (0, 1)$ such that, for each $\delta \geq \delta^*$, there exists $f_2^k \in F_2$ such that $v_1^{jk}(f_1, f_2^k, \delta) < v_1^{jk}$ for all $j \in \{1, \ldots, J\}$ and $f_1 \in F_1$.*

Proof. Let $B > 0$ be such that $|u_1^k(a)| \leq B$ for all $a \in A$ and $\varepsilon > 0$ be such that $2\varepsilon < \eta$ and $\varepsilon B + (1 - \varepsilon)(v_1^{jk} - \eta/2) < v_1^{jk}$ for each $j = 1, \ldots, J$. Let $S = \{y \in \mathbb{R}^J : y_j \leq v_1^{jk} - \eta \text{ for each } j = 1, \ldots, J\}$; since S is approachable with discounting by player 2, there exists $\delta^* \in (0, 1)$ such that, for every $\delta \geq \delta^*$, there exist $\tilde{f}_2 : \cup_{t=0}^{\infty} X^t \to \Delta(A_2)$ such that, for every $\tilde{f}_1 : \cup_{t=0}^{\infty} X^t \to \Delta(A_2)$,

$$P_{(\tilde{f}_1, \tilde{f}_2), \infty}\left(\left\{(x_1, x_2, \ldots) \in X^\infty : d\left((1-\delta)\sum_{t=1}^{\infty} \delta^{t-1} x_t, S\right) \geq \varepsilon\right\}\right) < \varepsilon.$$

Define $f_2 \in F_2$ by $f_2(H_0) = \tilde{f}_2(X^0)$ and, for each $h = (a_1, \ldots, a_t) \in H_t$ with $t > 0$, $f_2(h) = \tilde{f}_2(u_1^k(a_1), \ldots, u_1^k(a_t))$. Moreover, for each $f_1 \in F_1$, define $\tilde{f}_1 : \cup_{t=0}^{\infty} X^t \to \Delta(A_1)$ as follows: $\tilde{f}_1(X^0) = f_1(H_0)$ and, for each $x^t = (x_1, \ldots, x_t) \in X^t$ with $t > 0$,

$$\tilde{f}_1(x^t)[a_1] = \frac{\sum_{h \in H_t : u_1^k(h) = x^t} P_{(f_1,f_2),t}(h) f_1(h)[a_1]}{P_{(f_1,f_2),t}([x^t])},$$

where $u_1^k(h) = (u_1^k(a_1), \ldots, u_1^k(a_t))$ for each $h = (a_1, \ldots, a_t) \in H_t$ with $t > 0$ and $[x^t] = \{h \in H_t : u_1^k(h) = x^t\}$ (if $P_{(f_1,f_2),t}([x^t]) = 0$, then $\tilde{f}_1(x^t)$ is defined arbitrarily, e.g. $\tilde{f}_1(x^t)$ is uniform).

Given the above definition, it follows that

$$P_{(f_1,f_2),t}([x^t]) = P_{(\tilde{f}_1,\tilde{f}_2),t}(x^t)$$

for each $t \in \mathbb{N}$ and $x^t \in X^t$. Indeed, for each $t \in \mathbb{N}$, assuming that the equality holds for $t-1$, we have that

$$\begin{aligned} P_{(f_1,f_2),t}([x^t]) &= \sum_{h \in H_{t-1} : u_1^k(h) = x^{t-1}} P_{(f_1,f_2),t-1}(h) \sum_{a \in A} f_1(h)[a_1] f_2(h)[a_2] 1_{x_t}(u_1^k(a)) \\ &= P_{(f_1,f_2),t-1}([x^{t-1}]) \sum_{a \in A} \tilde{f}_1(x^{t-1})[a_1] \tilde{f}_2(x^{t-1})[a_2] 1_{x_t}(u_1^k(a)) \\ &= P_{(\tilde{f}_1,\tilde{f}_2),t-1}(x^{t-1}) \sum_{a \in A} \tilde{f}_1(x^{t-1})[a_1] \tilde{f}_2(x^{t-1})[a_2] 1_{x_t}(u_1^k(a)) = P_{(\tilde{f}_1,\tilde{f}_2),t}(x^t). \end{aligned}$$

The above then implies that, for each $T \in \mathbb{N}$,

$$\int_{X^T} \frac{1-\delta}{1-\delta^T} \sum_{t=1}^{T} \delta^{t-1} x_t \mathrm{d}P_{(\tilde{f}_1,\tilde{f}_2),T}(x^T) = \int_{A^T} \frac{1-\delta}{1-\delta^T} \sum_{t=1}^{T} \delta^{t-1} u_1^k(a_t) \mathrm{d}P_{(f_1,f_2),T}(a^T)$$

and, hence,

$$\int_{X^\infty} (1-\delta) \sum_{t=1}^{\infty} \delta^{t-1} x_t \mathrm{d}P_{(\tilde{f}_1,\tilde{f}_2),\infty}(x^\infty) = \int_{A^\infty} (1-\delta) \sum_{t=1}^{\infty} \delta^{t-1} u_1^k(a_t) \mathrm{d}P_{(f_1,f_2),\infty}(a^\infty).$$

Thus, it suffices to show that $\int_{X^\infty} (1-\delta) \sum_{t=1}^{\infty} \delta^{t-1} x_t \mathrm{d}P_{(\tilde{f}_1,\tilde{f}_2),\infty}(x^\infty) < v_1^k$. Let

$$C = \{x^\infty \in X^\infty : d\left((1-\delta) \sum_{t=1}^{\infty} \delta^{t-1} x_t, S\right) < \varepsilon\}$$

and note that $x^\infty \in C$ implies that, for all $j = 1, \ldots, J$ and some $y \in S$, $(1-\delta) \sum_{t=1}^{\infty} \delta^{t-1} x_t^j \leq y^j + \varepsilon < v_1^{jk} - (\eta - \varepsilon) < v_1^{jk} - \eta/2$. Hence, for each $j = 1, \ldots, J$,

$$\begin{aligned} &\int_{X^\infty} (1-\delta) \sum_{t=1}^{\infty} \delta^{t-1} x_t^j \mathrm{d}P_{(\tilde{f}_1,\tilde{f}_2),\infty}(x^\infty) \leq P_{(\tilde{f}_1,\tilde{f}_2),\infty}(C)(v_1^{jk} - \eta/2) + (1 - P_{(\tilde{f}_1,\tilde{f}_2),\infty}(C))B \\ &\leq \varepsilon B + (1-\varepsilon)(v_1^{jk} - \eta/2) < v_1^{jk}. \end{aligned}$$

This concludes the proof. ■

Combining Lemmas 10 and 11 yields Theorem 5.

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